Stability of the asymptotically hyperbolic manifolds and the heat kernel estimate

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Outline

Hyperbolic space vs conformal round sphere

Let us consider **Hyperbolic space** as a symmetric space

$$\mathrm{SO}^{\uparrow}(1, n+1)/\mathrm{SO}(n+1) = \mathbb{H}^{n+1}$$

and **the conformal round sphere** as the boundary at infinity of $\mathbb{H}^{n+1}$ and a homogeneous space of the parabolic geometry:

$$\mathrm{SO}^{\uparrow}(1, n+1)/\mathcal{H} = \mathbb{S}^n$$

where $\mathcal{H}$ is the group generated by rotations, scalings, and inversions on the Euclidean space $\mathbb{R}^n$. The identifications

$$\mathrm{SO}^{\uparrow}(1, n+1) \cong \mathrm{Isom}(\mathbb{H}^{n+1}) \cong \mathrm{Conf}(\mathbb{S}^n)$$

tell us how the **Lorentz group** acts on the hyperboloid in Minkowski spacetime and how the isometry group acts on the space of classes of equivalent geodesic rays in $\mathbb{H}^{n+1}$.

Hyperbolic space vs conformal round sphere

$(\mathbb{R}^{n+2}, g)$, $g = -dx_0^2 + |d\mathbf{x}|^2 + dx_{n+1}^2$

- **Hyperboloid** $\{(x_0, \mathbf{x}, x_{n+1}) | -x_0^2 + |x|^2 + x_{n+1}^2 = -1\}$ which is a orbit of $SO(1, n+1)$
- **Light cone** $\{(x_0, \mathbf{x}, x_{n+1}) | -x_0^2 + |x|^2 + x_{n+1}^2 = 0\}$ which is a orbit of $SO(1, n+1)$

AHE manifold vs conformal infinity

(X^{n+1}, g^+) is said to be asymptotically hyperbolic (AH in short) when (X^{n+1}, g^+) is conformally compact and, in addition, the sectional curvature goes to -1 at infinity and AH can induce a conformal structure on the boundary $\partial X^{n+1} = M^n$.

We like to impose the Einstein conditions

$$\mathrm{Ric}[g^+] = -ng^+$$

to make the association

$$\partial_\infty (X^{n+1}, g^+) = (M^n, [\hat{g}])$$

possibly canonical, in which (X^{n+1}, g^+) is said to be asymptotically hyperbolic Einstein (AHE in short).

The stability of the AHE

Let (M, g_+) be an AHE. We say that (M, g) is a stable AHE if for any other AHE (M,g) satisfying that $\|g - g_+\| < \varepsilon$ (Different $\|.\|$ will lead different stability), we always have

$$(M, g_+) \cong (M, g)$$

Introduction of Ricci flow

- **Ricci flow** The Rici flow is the geometric evolution equation in which one starts with a smooth Riemanian manifold $(\mathcal{M}^n, g_0)$ and evolves its metric by the equation there exists a smooth metric g in M satisfying
$$\frac{\partial}{\partial t} g = -2\mathrm{Rc}$$
where Rc denotes the Ricci tensor of the metric g.

Introduction of the Ricci flow (II)

- **The normalized Ricci flow**

$$\begin{cases} \frac{d}{dt} g(t) = -2\left(Ric_{g(t)} + ng(t)\right) \\ g(0) = g_0 \end{cases}$$

- **The relation to Ricci flow**

$$g^N(t) = e^{-2nt} g\left(\frac{1}{2(n-1)}\left(e^{2nt} - 1\right)\right)$$

- **The relation to the Einstein manifold** If the $g(\infty)$ exists, then $g(\infty)$ is a Einstein metric with $Ric(g(\infty)) = -ng(\infty)$

The relation between the stability of the AHE and the Ricci flow

In order to show the stability of one AHE (M, g_+), it suffices to show that for any $\|g - g_+\| < \varepsilon$, the normalized Ricci flow

$$\begin{cases} \dfrac{d}{dt} g(t) = -2\left(Ric_{g(t)} + ng(t)\right) \\ g(0) = g \end{cases}$$

have long time existence satisfying that

$$g(t) \to g_+ \quad \text{as } t \to \infty$$

Some important results

Theorem (R. Bamler 2015)

Let $(M, \overline{g})$ be either $\mathbb{H}^n$ for $n \geq 3$ or $\mathbb{CH}^{2n}$ for $n \geq 2$, choose a basepoint $x_0 \in M$ and let $r = d(\cdot, x_0)$ denote the radial distance function. There is an $\varepsilon_1 > 0$ and for every $q < \infty$ an $\varepsilon_2 = \varepsilon_2(q) > 0$ such that the following holds: If $g_0 = \overline{g} + h$ and $h = h_1 + h_2$ satisfies

$$|h_1| < \frac{\varepsilon_1}{r+1} \quad \text{and} \quad \sup_M |h_2| + \left(\int_M |h_2|^q \, dx \right)^{1/q} < \varepsilon_2.$$

Then the normalized Ricci flow exists for all time and we convergence $g_t \longrightarrow \overline{g}$ in the pointed Cheeger-Gromov sense.

Some important results

Theorem (J.Qing and S)

Let (M^{n+1}, g_+) be an asymptotically hyperbolic Einstein manifolds with nondegeneracy $\lambda > 0$ and regularity $C^{2,\alpha}$ and (M^{n+1}, g) be another asymptotically hyperbolic Einstein manifolds. Then, for any $\delta \in (0, n)$, there exists $\epsilon_0(\lambda) > 0$, such that if $|g - g_+| \leq \epsilon_0 e^{-\delta d(x_0, x)}$, Then the Ricci DeTurck flow with the initial g has the long time existence and

$$\lim_{t \to \infty} |g - g_+|_{0,0,\delta} = 0$$

Our goal

Theorem (Our Goal)

Let $M = B^{n+1}$ be a ball wiht $n \geq 3$ and $\hat{h}$ the standard metric on the sphere S^n and $g_{\mathbb{H}}$ be the standard hyperbolic metric on B^{n+1}. For any asymptotically hyperbolic Einstein manifold (M, g) with nonpositive sectional curvature and a defining function ρ such that $\hat{g} = \rho^2 g|_{\partial M}$ is sufficiently close to $\hat{h}$ in $C^{2,\alpha}$ norm, for some $0 < \alpha < 1$ and g is sufficiently close to $g_{\mathbb{H}}$ in the sense of C^0. And choose a basepoint $x_0 \in M$ and let $r = d(., x_0)$ denote the radial distance function. There is an $\epsilon > 0$ such that the following holds: If $g_0 = g + h$ satisfies

$$|h| < \frac{\epsilon}{r+1}$$

then the normalized Ricci flow exists for all time and we have convergence $g_t \to g$ in the pointed Cheeger-Gromov sense.

Short time existence

Theorem (Shi 1989)

Let $(M, g_{ij}(x))$ be an n -dimensional complete noncompact Riemannian manifold with its Riemannian curvature tensor $\{R_{ijkl}\}$ satisfying

$$|R_{ijkl}|^2 \leq k_0 \quad \text{on } M$$

where $0 < k_0 < +\infty$ is a constant. Then there exists a constant $T(n, k_0) > 0$ depending only on n and k_0 such that the evolution equation

$$\frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t) \text{ on } M$$

$$g_{ij}(x,0) = g_{ij}(x) \quad \forall x \in M$$

has a smooth solution $g_{ij}(x,t) > 0$ for a short time $0 \leq t \leq T(n, k_0)$,.

The Long time existence of the Ricci flow

The following metric flow is called the **normalized Ricci-DeTurck flow**

$$\frac{\partial}{\partial t}g_{ij} = -2R_{ij}(g(t)) + \nabla_i W_j + \nabla_j W_i - 2ng_{ij}$$

where $W_j = g^{ll_1}g_{jk}(\Gamma^k_{ll_1}(g(t)) - \Gamma^k_{ll_1}(g_+))$ and ∇ is the covariant derivative with respect to $g(t)$.

Moreover, there exists a 1-parameter family of maps $\varphi_t : M \to M$ satisfying that

$$\frac{\partial}{\partial t}\varphi_t(p) = -W(\varphi_t, t) \quad \varphi_0 = id$$

such that $\tilde{g}(t) := \varphi_t^*(g(t))$ is a solution a the normalized Ricci flow.

Linearization

- **The Linearization of normalized Ricci-deTurk flow** $h_{ij}(t,x) = g_{ij}(t,x) - g_{+ij}(x)$. Then the Ricci-DeTurck flow is equivalent to the following flow

$$\frac{\partial}{\partial t} h_{ij} = \tilde{\Delta} h_{ij} - 2\tilde{R}_{jlil_2} h^{ll_2} - \tilde{R}_{il_2} h_j^{l_2} + \tilde{R}_{jl_2} h_i^{l_2} - 2n h_{ij} \\ - 2(\tilde{R}_{ij} + n\tilde{g}_{ij}) + Q_{ij}(t,x)$$

where $\tilde{\Delta}$, $\tilde{R}$ is respect to $\tilde{g}_{ij} = g_{+ij}$ and

$$Q_{ij}(t,x) = \tilde{g} * \tilde{g}^{-1} * \tilde{\nabla} h * \tilde{\nabla} h + \tilde{g} * \tilde{g}^{-1} * \tilde{\nabla}^2 h * h$$

- The long time existence depends on the estimate of the heat kernel.

The heat kernel estimate

In order to show our goal, we need the estimates for heat kernel

$$\partial_t k_t = \Delta k_t + R(k_t) \quad \textit{and} \quad k_t \to \delta_{p_0} id_{E_{p_0}} \quad \textit{as } t \to 0 \qquad (1)$$

where $(k_t)_{0<t<T} \in C^\infty(M;E) \otimes E_{p_0}$ and $E = Sym^2 T^* M$ and $R(h)_{il} = -2\tilde{R}_{jlil_2} h^{ll_2} - \tilde{R}_{il_2} h_j^{l_2} + \tilde{R}_{jl_2} h_i^{l_2}$. Following the method of R.Bamler, we see that it is sufficient to show that

$$\|k_t\|_{L^1(M)} \leq C \quad \|k_t\|_{L^2(M)} \leq C \exp\left(\lambda_B t\right) \qquad \text{for } t > 0$$

The heat kernel estimate(II)

In the light of the result of Davies and Mandouvalos, we can show if we have the following heat kernel estimate

$$|k_t| \leq t^{-(n+1)/2} \exp\left(-\frac{n^2 t}{4} - \frac{r^2}{4t} - \frac{nr}{2}\right) \cdot (1+r+t)^{n/2-1}(1+r),$$

Then we have

$$\|k_t\|_{L^1(M)} \leq C \quad \|k_t\|_{L^2(M)} \leq C \exp\left(\lambda_B t\right) \qquad \text{for } t > 0.$$

An important result

Theorem (X.Chen and A.Hassell 2017)

Let (M, g) be an $(n+1)$ dimensional asymptotically hyperbolic Cartan-Hadamard manifold with no eigenvalues and no resonance at the bottom of the spectrum. Let $k(t, z, z')$ be the heat kernel on (M, g). Then $k(t, z, z')$ is equivalent to the Davies-Mandouvalos quantity, i.e. bounded above and below by multiples of

$$t^{-(n+1)/2} \exp\left(-\frac{n^2 t}{4} - \frac{r^2}{4t} - \frac{nr}{2}\right) \cdot (1+r+t)^{n/2-1}(1+r)$$

uniformly over all times $t \in (0, \infty)$ and distances $r = d(z, z') \in (0, \infty)$.

The idea of X.Chen and A.Hassell

The realtion between the heat operator and resolvent. In order to find the heat kernel it suffices to make sense $e^{-\Delta t}$. Let A be a self-adjoint operator on $\mathcal{H}$. And let $h : \mathbb{R} \to \mathbb{C}$ bounded and continuous. Then, we can define

$$h(A) = \int_{\mathbb{R}} h(\lambda) dP(\lambda)$$

where $dP(\lambda)$ is the spectrum measure of the operator A.

(II)

Theorem (Stone)

The spectral projectors associated with a self-adjoint operator A are expressed in terms of the resolvent by,

$$\frac{1}{2}\left(P_{[\alpha,\beta]} + P_{(\alpha,\beta)}\right) = \int_{\alpha}^{\beta} d\Pi(\lambda)$$

where $d\Pi$ is the spectrum measure.

(III)

Now, Let (M^{n+1}, g) be an asymptotically hyperbolic manifold. If A is a operator on $\mathrm{Sym}^2(T^*M)$ with all its spectrum in $[\frac{n^2}{4}, \infty)$ and no eigenvalue at $\frac{n^2}{4}$. Then, consider the operator $B = A - \frac{n^2}{4}$. And take $h(x) = e^{-tx}$. Then consider the following operator

$$h(B) = \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} h(\lambda) d\Pi(\lambda) = \lim_{\varepsilon \to 0} \int_0^{\infty} e^{-t\lambda} d\Pi(\lambda)$$

where

$$d\Pi(\lambda) := \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \left[(B - \lambda - i\varepsilon)^{-1} - (B - \lambda + i\varepsilon)^{-1} \right] d\lambda$$

(IV)

Let $\lambda = s^2$ and $s = a + bi$

$$d\Pi\left(a^2\right) = d\Pi\left(\mathrm{Re}\left(s^2\right)\right) = \frac{1}{2\pi i}\lim_{b\to 0}\left[\left(B - s^2\right)^{-1} - \left(B - \bar{s}^2\right)^{-1}\right] 2ada$$

$$d\Pi\left(a^2\right) = \frac{1}{2\pi i}\lim_{b\to 0}\left[R(s) - R(\bar{s})\right] 2ada = \frac{1}{2\pi i}\lim_{b\to 0}\left[R(a+bi) - R(a-bi)\right.$$

where $R(s) = (B - s^2)^{-1}$. We have have that

$$h(B) = \lim_{b\to 0^+} -\frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{-ta^2}\left[R(a-bi)\right] 2ada$$

(V)

Therefore,

$$\begin{aligned} h(A) =& h(B+\frac{n^2}{4}) = \lim_{\varepsilon\to 0}\int_0^\infty e^{-t(\lambda+\frac{n^2}{4})}d\Pi(\lambda) \\ =& \lim_{\varepsilon\to 0} e^{-t\frac{n^2}{4}}\int_0^\infty e^{-t\lambda}d\Pi(\lambda) = e^{-t\frac{n^2}{4}}h(B) \\ =& e^{\frac{-n^2}{4}t}\lim_{b\to 0^+} -\frac{1}{2\pi i}\int_{-\infty}^\infty e^{-ta^2}\left[R(a-bi)\right]2ada \end{aligned}$$

Moreover, we can show that

$$\frac{d}{dt}h(A)(u) = Ah(A)u$$

(VI)

The result of X. Chen and A. Hassell relies on the following result

Theorem (R.Melrose, A.Sa Barreto and A.Vasy 2014)

Assume that (X, g) is an asymptotically hyperbolic Cartan-Hadamard manifold with no eigenvalues and no resonance at the bottom of the spectrum. Let r denote geodesic distance on $X \times X$. Then the resolvent, $R(\lambda) := \left(\Delta_X - n^2/4 - \lambda^2\right)^{-1}$ is analytic in a neighbourhood of the closed lower half plane $\operatorname{Im}\lambda \leq 0$, and satisfies in this region of the λ -plane and for $r(1+|\lambda|) \geq 1$.

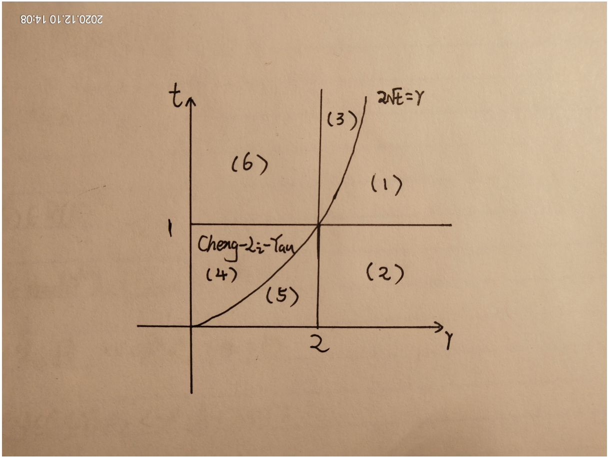
$$R(\lambda)\left(z, z'\right) = e^{-i\lambda r} R_{od}(\lambda)\left(z, z'\right), \quad r = d\left(z, z'\right)$$

(VII)

Theorem (R.Melrose, A.Sa Barreto and A.Vasy 2014)

In particular, $R_{od}(\lambda)$ *is a kernel bounded pointwise by a multiple of* $(r(1+|\lambda|))^{n/2-1}r^{-n+1} = r^{-n/2}(1+|\lambda|)^{n/2-1}$ *for* $r \leq C$*, and,* $e^{-nr/2}(1+|\lambda|)^{n/2-1}$ *for* $r \geq C$*.*

(VIII)



(Cheng-Li-Yau)

Theorem (Cheng-Li-Yau)

Let M be a complete non-compact Riemannian manifold whose sectional curvature is bounded from below and above. For any constant $C > 4$, there exists C_1 depending on $C, T, z \in M$, the bounds of the curvature of M so that for all $t \in [0, T]$ the heat kernel $H(t, z, z')$ obeys

$$h(t, r(z, z')) \leq \frac{C_1(C, T, z)}{\left|B_{\sqrt{t}}(z)\right|} \exp\left(-\frac{r^2(z, z')}{Ct}\right)$$

where $r(z, z')$ is the geodesic distance on M.

Cheng-Li-Yau

Form the above theorem, we see that in our asymptotically hyperbolic manifold (X^{n+1}, g_+), its heat kernel, in this region, has the following estimate

$$h(t, r(z, z')) \leq \frac{C}{t^{\frac{n+1}{2}}} \leq h_{\mathbb{H}^{n+1}}(t, r)$$

where

$$h_{\mathbb{H}^{n+1}}(t, r) \sim \frac{1}{t^{\frac{n+1}{2}}} e^{-\frac{n^2}{4}t} e^{-\frac{nr}{2}} e^{\frac{r^2}{4t}} (1 + r + t)^{\frac{n}{2}-1}(1 + r)$$

Region (6)

Theorem (X.Chen and A.Hassell 2016)

Suppose (X, g) *is an* $n+1$ *-dimensional asymptotically hyperbolic CartanHadamard manifold with no resonance at the bottom of the continuous spectrum and denote the operator* $\sqrt{(\Delta_X - n^2/4)_+}$ *by* P. *The Schwartz kernel of the spectral measure* $dE_P(\lambda)$ *satisfies bounds*

$$\left|dE_P(\lambda)\left(z, z'\right)\right| \leq \begin{cases} C\lambda^2, & \text{if} \quad \lambda \leq 1 \\ C\lambda^n & \text{if} \quad \lambda \geq 1 \end{cases}$$

Thank you for your time!