Normalized Ricci flow on asymptotically hyperbolic manifolds

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Abstract. In this paper, we investigate the behavior of the normalized Ricci flow on asymptotically hyperbolic manifolds. We show that the normalized Ricci flow exists globally and converges to an Einstein metric when starting from a non-degenerate and sufficiently Ricci pinched metric. More importantly, motivated by [QSW2013] [Ba2015], we also establish the regularity of conformal compactness of the normalized Ricci flow towards time infinity. Therefore we are able to fully recover the existence results in [GL1991] [Le2006] [Bi1999] of conformally compact Einstein metrics with conformal infinities thich are perturbation of that of given non-degenerate conformally compact Einstein.

Contents

1 Introduction

Since the seminal work of Fefferman and Graham [FG1985] there have been great interests in the study of conformally compact Einstein metrics. Lately the use of conformally compact Einstein manifolds in the so-called AdS/CFT correspondence in string theory proposed as a promising quantum theory of gravity have accelerated developments of the study of conformally compact Einstein manifolds. As it was foreseen in [FG1985], the study of conformally compact Einstein manifolds now becomes one of the most active research area in conformal geometry. But the existence of conformally compact Einstein metrics remains to be a challenging open problem in large.

In this paper we study the normalized Ricci flows on asymptotically hyperbolic manifolds and use normalized Ricci flows to construct conformally compact Einstein metrics. We recall that Ricci flow starting from a metric g_0 on a manifold M^n is a family of metrics $g(t)$ that satisfies the following:

$$\begin{cases} \dfrac{d}{dt}g(t) = -2\,\mathrm{Ric}_{g(t)} \\ g(0) = g_0 \end{cases}$$

We then consider the normalized Ricci flow as follows:

$$\begin{cases} \dfrac{d}{dt}g(t) = -2\left(\mathrm{Ric}_{g(t)} + ng(t)\right) \\ g(0) = g_0 \end{cases}$$

It is easily seen that the above two equations are equivalent. In fact explicitly

$$g^N(t) = e^{-2nt} g\left(\frac{1}{2n}\left(e^{2nt} - 1\right)\right)$$

solves the second equation if and only if g(t) solves the first equation.

Naturally one initial step is to study normalized Ricci flows starting from metrics that are close to be Einstein. Such questions on compact manifolds were studied in [Ye1993], where it was observed that the normalized Ricci flow exists globally and converges exponentially to an Einstein metric if the initial metric g_0 is sufficiently Ricci pinched and is non-degenerate. There are also several works in the non-compact cases. In [LY2010], the stability of the hyperbolic space under the normalized Ricci flow was established. This stability result on the hyperbolic space in [LY2010] later is improved and extended in [Ba2015] [Ba2014] [SSS2010] [Su2009].

To be more precise we say a metric g on a manifold $\mathcal{M}^n$ is ϵ-Einstein if

$$\|h_g\| \leq \epsilon$$

on $\mathcal{M}^n$, where the Ricci pinching curvature $h_g = Ric_g + (n-1)g$. The non-degeneracy of a metric is defined to be the first L^2 eigenvalue of the linearization of the curavture tensor h as follows:

$$\lambda = \inf \frac{\int_{\mathcal{M}} \langle (\Delta_L + 2(n-1))\, u_{ij}, u_{ij} \rangle}{\int_{\mathcal{M}} \|u\|^2}$$

where the infimum is taken among symmetric 2-tensors u such that

$$\int_{\mathcal{M}} \left(|\nabla u|^2 + |u|^2\right) dv < \infty$$

and Δ_L is **Lichnerowicz Laplacian** on symmetric 2-tensors.

We first, based on the ideas in [Ye1993] [Ba2015], obtain the following global existence and convergence theorem of the normalized Ricci flow on non-compact manifolds. The reason that we consider the curvature flow is that this flow is strictly elliptic flow which is easier than directly considering the Ricci DeTurck flow. In the setting of Ricci DeTurck flow, we still need to consider the long time existence and convergence of harmonic map flow.

Theorem 1.1 *Let (M^{n+1}, g_+) be an asymptotically hyperbolic manifolds with nondegeneracy $\lambda > 0$, regularity $C^{2,\alpha}$ and $n \geq 4$. Then, for any $\delta \in (0, n)$, there exists $\epsilon_0(\lambda, k_1) > 0$ such that if $|h|_{0,0,\delta;M} \leq \epsilon_0$, the solution of the normalized Ricci flow $g(t,x)$ has long time existence and $g(t,x)$ converges to an Einstein manifold in the sense of C^2_δ norm. Moreover, the limit metric is an Asymptotically hyperboli Einstein metric with the same conformal infinity.*

The theorem 1.1 actually is a generalization of the theorem 4.1 in [QSW2013]. In [QSW2013], they require the weight δ satisfied the following

$$\delta \in \left(\frac{n}{2} - \min\left\{\sqrt{\lambda}, \sqrt{\frac{n^2}{4} - 2}\right\}, \frac{n}{2} + \sqrt{\frac{n^2}{4} - 2}\right)$$

In this paper, we only require that

$$\delta \in \left(\frac{n}{2} - \sqrt{\frac{n^2}{4}}, \frac{n}{2} + \sqrt{\frac{n^2}{4}}\right) = (0, n)$$

The reason that we can modified the term $\sqrt{\frac{n^2}{4} - 2}$ into $\sqrt{\frac{n^2}{4}}$ is that we use a more precise isomorphism theorem of Laplacian operator on weighted space

(Theorem C [Le2006]) instead of the maximal principal for L^2 norm of h.

Moreover, once we have the long time existence and convergence of normalized Ricci flow, we can derive the following stability theorem of asymptotically hyperbolic manifolds.

Theorem 1.2 *Let (M^{n+1}, g_+) be an asymptotically hyperbolic Einstein manifolds with nondegeneracy $\lambda > 0$, regularity $C^{2,\alpha}$ and $n \geq 4$. Let g be another asymptotically hyperbolic metric on M^{n+1}. Then, for any $\delta \in (0, n)$, there exists $\epsilon_0(\lambda) > 0$, such that if $|g - g_+| \leq \epsilon_0 e^{-\delta d(x_0,x)}$, Then the Ricci DeTurck flow with the initial g has the long time existence and*

$$\lim_{t\to\infty} |g - g_+|_{0,0,\delta} = 0$$

For the stability result of hyperbolic space $M^{n+1} = \mathbb{H}^{n+1}$, Schulze, Schnurer and Simon ([SSS2010]) have shown stability of $n \geq 3$ for every perturbation $|g - g_{\mathbb{H}^{n+1}}|_{L^\infty}$ is bounded by a small constant depending on $\|g - g_{\mathbb{H}^{n+1}}\|_{L^2}$.

While Li and Yin ([LY2010]) have shown a stability result of $n \geq 2$ for the Riemannian curvature approaches the hyperbolic curvature like $\epsilon_1(\delta)e^{-\delta d(x_0,x)}$

Furthermore, Bamler ([Ba2015]) have shown stability of $n \geq 2$ for the perturbation $|g - g_{\mathbb{H}^{n+1}}| = h_1 + h_2$ for which

$$|h_1| \leq \frac{\epsilon_1}{d(x_0,x)+1} \quad \text{and} \quad \sup_M |h_2| + \left(\int_M |h_2|^q\right)^{\frac{1}{q}} \leq \epsilon_2$$

for every $q < \infty$.

It easy to see that the stability result of [Ba2015] just implies that the stability result of [SSS2010].

For the theorem 1.2, if we take g_+ is the standard hyperbolic metric, then this stability result is implied by the stability result of [Ba2015].

By the theorem 1.1, we can fully recover the perturbation existence results in [GL1991] [Le2006] [Bi1999]. The idea is to construct an asymptotically hyperbolic metric with prescribed boundary which satisfying the condition of theorem 1.1. Then we apply the theorem 1.1 to get the asymptotically hyperbolic Einstein metric with this boundary.

Theorem 1.3 *Let (M^n, g_+), be a conformally compact Einstein manifold of regularity C^2 with a smooth conformal infinity $(\partial M, [\hat{g}])$. And suppose that the non-degeneracy of g satisfies*

$$\lambda > 0$$

Then, for any smooth metric $\hat{h}$ on ∂M, which is sufficiently $C^{2,\alpha}$ close to some $\hat{g} \in [\hat{g}]$ for any $\alpha \in (0,1)$, there is a conformally compact Einstein metric on M which is of C^2 regularity and with the conformal infinity $[\hat{h}]$

Our paper is organized as follows: In section 2.1, we first introduce the normalized Ricci flow and its curvature flow which is to prove the long time existence and convergence of normalized Ricci flow. Then we introduce the Ricci DeTurck flow and its linearization which is to show the stability result of theorem 1.2. In section 2.2, we introduce some basic concepts of asymptotticallly hyperbolic manifolds and Mobius chart which are important to do the parabolic estimate on the asymptotically hyperbolic manifolds. In section 2.3, we just introduce the interior parabolic estimate on weighted space (The corresponding elliptic estimate can be seen in Lemma 4.8 of [Le2006]). In section 2.4, we just generalized the result short time existence of Ricci flow [Shi1989] and [Mi2002] into the weighted space that is to say the Ricci flow preserve the decay of metric for a short time. The idea is from Lemma 4.3 in [QSW2013] which is a generalized maximal principal (Lemma 4.2 in [QSW2013]). In section 2.5, we use the Hille-Yosida thoerem about semigroup to show that the isomorphism theorem [GL1991] [Bi1999] is equivalent to the exponential decay of the heat kernel of the linear heat equation. In section 3, we make use of the method of [Ba2015] plus a little tricky linearization method to get the long time existence and convergence of the normalized Ricci flow. In the section 4, we recall the metric expansions in [FG1985] for conformally compact Einstein metric and apply normalized Ricci flows to reproduce perturbation existence results in [GL1991] [QSW2013] [Bi1999].

2 Preliminary

In this section, we will review some basic result of normalized Ricci flow and parabolic equation on asymptotically hyperbolic manifolds.

2.1 Curvature flow and Ricci DeTurck flow and its linearizations

Let $g(t)$ be a family of metrics on the same manifolds M^{n+1} satisfying the normalized Ricci flow

$$\begin{cases} \dfrac{\partial}{\partial t} g(t) = -2\left(\mathrm{Ric}_{g(t)} + n g(t)\right) \\ \quad g(0) = g_+ \end{cases}$$

Let $h(t) = Ric_{g(t)} + ng(t)$. Then, we can get the evolution equation of $h(t)$ as following

$$\frac{\partial}{\partial t} h_{il} = \Delta_L h_{il} - 2n h_{il}$$

where Δ_L is the **Lichnerowicz Laplacian** operator defining as following

$$\Delta_L h_{il} = \Delta h_{il} - g^{jk_1} R_{lj} h_{k_1 i} - g^{jk_1} R_{ij} h_{k_1 l} + 2 g^{jk_1} g^{i_1 i_2} R_{i i_2 l j} h_{k_1 i_1}$$

Moreover, we can also write the above as

$$\frac{\partial}{\partial t}h_{il} = \Delta_{L(g(t-l))}h_{il} - 2nh_{il} + Q$$

where

$$\begin{aligned} Q =& [\Delta_{L(g(t))} - \Delta_{L(g(t-l))}]h_{il} \\ =& g(t) * g(t-l) * [\tilde{\nabla} g(t) * \tilde{\nabla} g(t) + g(t) * (\tilde{\nabla}^2 g(t) + R(g(t-l)))] * h \\ &+ g(t) * g(t-l) * [\tilde{\nabla} g(t)] * \tilde{\nabla} h \end{aligned}$$

where $\tilde{\nabla}$ is with respect to $g(t-l)$.

The following metric flow is called the **normalized Ricci-DeTurck flow**

$$\frac{\partial}{\partial t}g_{ij} = -2R_{ij}(g(t)) + \nabla_i W_j + \nabla_j W_i - 2(n-1)g_{ij}$$

where $W_j = g^{ll_1} g_{jk}(\Gamma^k_{ll_1}(g(t)) - \Gamma^k_{ll_1}(g(0)))$ and ∇ is the covariant derivative with respect to $g(t)$.

Linearization : Let $h_{ij}(t,x) = g_{ij}(t,x) - g_{ij}(0,x)$. Then the Ricci-DeTurck flow is equivalent to the following flow

$$\frac{\partial}{\partial t}h_{ij} = \tilde{\Delta}_L h_{ij} - 2(n-1)h_{ij} - 2(\tilde{R}_{ij} + (n-1)\tilde{g}_{ij}) + Q'_{ij}(t,x)$$

where $\tilde{\Delta}_L$ and $\tilde{R}$ are the **Lichnerowicz** Laplacian operator and Ricci curvature with respect to $g(0) = \tilde{g}$ and the high order term Q is

$$Q'_{ij}(t,x) = g * g^{-1} * \tilde{g} * \tilde{g}^{-1} * \tilde{\nabla} h * \tilde{\nabla} h + g * g^{-1} * \tilde{g} * \tilde{g}^{-1} * \tilde{\nabla}^2 h * h$$

For details of the computation, see the appendix.

2.2 Asymptotically hyperbolic manifolds and its Mobius charts

In this section, we will introduce the asymptotically hyperbolic manifolds and Mobius chart. In the Mobius chart of asymptotically hyperbolic manifolds, the metric can be uniformly bounded (See Lemma 2.1) and approaching the standard hyperbolic metric as approaching the boundary. Therefore, we can get a pretty good globally elliptic and parabolic estimate. Most content of this section is from [Le2006].

2.2.1 Asymptotically hyperbolic manifolds

In order to define the asymptotically hyperbolic manifolds, we need to first introduce the conformally compact manifold. Defining function is the key in these concepts.

Definition 2.1 (Defining function) *Let $\bar{M}$ be a smooth, compact, $(n+1)$ -dimensional manifold-with-boundary, $n \geq 1$, and M its interior. A defining function will mean a function $\rho : \bar{M} \to \mathbb{R}$ of class at least C^1 that is positive in M, vanishes on ∂M, and has nonvanishing differential everywhere on ∂M.*

Definition 2.2 (Conformal compactness) *A Riemannian metric g on M is said to be conformally compact of class $C^{l,\beta}$ for a nonnegative integer l and $0 \leq \beta < 1$ if for any smooth defining function ρ, the conformally rescaled metric $\rho^2 g$ has a $C^{l,\beta}$ extension, denoted by $\bar{g}$, to a positive definite tensor field on $\bar{M}$.*

Remark 2.1 *For such a metric g, the induced boundary metric $\widehat{g} := \bar{g}|_{T\partial M}$ is a $C^{l,\beta}$ Riemannian metric on ∂M whose conformal class $[\widehat{g}]$ is independent of the choice of smooth defining function ρ; this conformal class is called the **conformal infinity** of g.*

Definition 2.3 (Asymptotically hyperbolic manifolds) *If g is conformally compact of class $C^{l,\beta}$ with $l \geq 2$, and $|d\rho|^2_{\bar{g}} = 1$ on ∂M, we say g is **asymptotically hyperbolic of class** $C^{l,\beta}$ and the corresponding manifold is called **asymptotically hyperbolic manifold**.*

We begin by choosing a covering of a neighborhood of ∂M in $\bar{M}$ by finitely many smooth coordinate charts (Ω, Θ), where each coordinate map Θ is of the form $\Theta = (\theta, \rho) = \left(\theta^1, \ldots, \theta^n, \rho\right)$ and extends to a neighborhood of $\bar{\Omega}$ in $\bar{M}$. Throughout this monograph, we will use the Einstein summation convention, with Roman indices $i, j, k, \ldots$ running from 1 to $n+1$ and Greek indices $\alpha, \beta, \gamma, \ldots$ running from 1 to n. Therefore, we can write $\left(\theta^1, \ldots, \theta^n, \rho\right)$ as θ^i if we think of ρ as θ^{n+1}.

We fix once and for all finitely many such charts covering a neighborhood W of ∂M in $\bar{M}$. We will call any of these charts "**background coordinates**" for $\bar{M}$. Take a local background coordinate (θ, ρ). Define $H_c(p)$ as the following set

$$Z_c(p) \triangleq \{(\theta, \rho) : |\theta - \theta(p)| < c, 0 \leq \rho < c\}$$

And define the set A_c as following

$$A_c \triangleq \{p \in W : \exists \text{ backgroud local coordinate chart } (U, \theta^i) \text{ such that } Z_c(p) \subset U\}$$

We see that for $c_1 \leq c_2$, we have $A_{c_2} \subset A_{c_1}$. And by the compactness $\bar{M}$, there exist c_0 such that A_{c_0} forms a neighborhood of ∂M. Now, we will define the Mobius charts based on these background coordinates and the standard coordinate of hyperbolic space.

In the upper half-space model, we regard hyperbolic space as the open upper half-space

$$\mathbb{H} = \mathbb{H}^{n+1} \triangleq \{(x^1, \cdots, x^n, y) \subset \mathbb{R}^{n+1} : y > 0\}$$

endowed with the hyperbolic metric $\breve{g} = y^{-2} \sum_i \left(dx^i\right)^2$.

For any $r > 0$, we let $B_r \subset \mathbb{H}$ denote the hyperbolic geodesic ball of radius r about the point$(x, y) = (0, 1)$

$$B_r = \{(x, y) \in \mathbb{H} : d_{\breve{g}}((x, y), (0, 1)) < r\}$$

Then

$$B_r \subset \left\{(x, y) : |x| < \sinh r, e^{-r} < y < e^r\right\}$$

where $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^n$.

If p_0 is any point in $A_{c_0/8}$, choose such a background chart containing p_0, and $\{(\theta, \rho) : |\theta - \theta(p_0)| \leq c_0, 0 < \rho < c_0\}$ and define a map $\Phi_{p_0} : B_2 \to M$, called a **Möbius chart** centered at p_0, by

$$(\theta, \rho) = \Phi_{p_0}(x, y) = (\theta_0 + \rho_0 x, \rho_0 y)$$

where (θ_0, ρ_0) are the background coordinates of p_0. Therefore, we see that

$$|\theta - \theta_0| \leq \rho_0 x \leq \rho_0 \sinh(2) \leq 4\rho_0 \qquad \rho \leq \rho_0 e^2 \leq 8\rho_0$$

Since $p_0 \in A_{c_0/8}$, $\rho_0 \leq c_0/8$. Therefore,

$$\Phi(B_2) \subset \{(\theta, \rho) : |\theta - \theta(p)| \leq c_0, 0 < \rho < c_0\}$$

is still contained in the same background local coordinate.

We also choose finitely many smooth coordinate charts $\Phi_i : B_2 \to M$ such that the sets $\{\Phi_i(B_2)\}$ cover a neighborhood of $M \backslash A_{c_0/8}$,. For consistency, we will also call these "**Mobius charts**." Therefore, we have a Mobius charts covering

$$\{\Phi_i(B_2), \Phi_i\}_{i=1}^N \cup \{\Phi_{p_0}(B_2), \Phi_{p_0}\}_{p_0 \in A_{c_0/8}}$$

For simplicity, we just write is as

$$\{\Phi_{p_i}(B_2), \Phi_{p_i}\}_{p_i \in M}$$

where $\Phi_{p_i}(\mathbf{0}, 1) = p_i$.

The following lemma shows the uniformly bounded of the Mobius coordinate.

Lemma 2.1 (Lemma 2.1 [Le2006]) *There exists a constant $C > 0$ such that if $\Phi_{p_0} : B_2 \to M$ is any y, Möbius chart,*

$$\left\|\Phi_{p_0}^* g - \breve{g}\right\|_{C^{l,\beta}(B_2)} \leq C$$
$$\sup_{B_2} \left|\left(\Phi_{p_0}^* g\right)^{-1} \breve{g}\right| \leq C$$

(The Hölder and sup norms in this estimate are the usual norms applied to the components of a tensor in coordinates; since $\bar{B}_2$ is compact, these are equivalent to the intrinsic Hölder and sup norms on tensors with respect to the hyperbolic metric.

2.2.2 Weighted Holder spaces

In this section, we will define the weighted Holder space on the asymptotically hyperbolic manifolds by the Mobius coordinate. Most of the content of this section is from [Le2006].

Throughout this section, we assume $\bar{M}$ is a connected smooth $(n+1)$ -manifold, g is a metric on M that is asymptotically hyperbolic of class $C^{l,\beta}$, with $l \geq 2$ and $0 \leq \beta < 1$, and ρ is a fixed smooth defining function for ∂M. (It is easy to verify that choosing another smooth defining function will replace the norms we define below by equivalent ones, and will leave the function spaces unchanged.)

A **geometric tensor bundle** over $\bar{M}$ is a subbundle E of some tensor bundle $T^{r_1}_{r_2}\bar{M}$ (tensors of covariant rank r_1 and contravariant rank r_2) associated to a direct summand (not necessarily irreducible) of the standard representation of $\mathrm{O}(n+1)$ (or $\mathrm{SO}(n+1)$ if M is oriented) on tensors of type $\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$ over $\mathbb{R}^{n+1}$. We will also use the same symbol E to denote the restriction of this bundle to M.

Definition 2.4 (Holder space) *Let (M^{n+1}, g) be an asymptotically hyperbolic manifold with boundary regularity $C^{l,\beta}$, $l \geq 2$. Let α be a real number such that $0 \leq \alpha < 1$, and let k be a nonnegative integer such that $k + \alpha \leq l + \beta$. For any tensor field u with locally $C^{k,\alpha}$ coefficients, define the norm $\|u\|_{k,\alpha}$ by*

$$\|u\|_{k,\alpha} := \sup_{\Phi} \|\Phi^* u\|_{C^{k,\alpha}(B_2)}$$

*where $\|v\|_{C^{k,\alpha}(B_2)}$ is just the usual Euclidean Hölder norm of the components of v on $B_2 \subset \mathbb{H}$, and the supremum is over all Möbius charts defined on B_2. Let $C^{k,\alpha}(M;E)$ be the space of sections of E for which this norm is finite. This space is called **Holder space**.*

Definition 2.5 (Weighted Holder spaces) *The **Weighted Hölder spaces** are defined for $\delta \in \mathbb{R}$ by*

$$C^{k,\alpha}_{\delta}(M;E) := \rho^{\delta} C^{k,\alpha}(M;E) = \left\{\rho^{\delta} u : u \in C^{k,\alpha}(M;E)\right\}$$

with norms

$$\|u\|_{k,\alpha,\delta} := \left\|\rho^{-\delta} u\right\|_{k,\alpha}$$

Remark 2.2 *If $U \subset M$ is a subset, the restricted norms are denoted by $\|\cdot\|_{k,\alpha,\delta;U}$, and the space $C^{k,\alpha}_{\delta}(U;E)$ are the spaces of sections over U for which these norms are finite.*

The following lemma just show that the above Holder norm actually is equivalent to the usual intrinsic C^k norm $\sum_{0\le i\le k}\sup_M|\nabla^i u|$ for $0\le k\le l$.

Lemma 2.2 (Lemma 3.4 [Le2006]) *Let (M^{n+1},g) be an asymptotically hyperbolic manifold with boundary regularity $C^{l,\beta}$, $l\ge 2$. Let u be a locally integrable section of a tensor bundle E over an open subset $U\subset M$ If $0\le\alpha<1$ and $0<k+\alpha\le l+\beta, u\in C^{k,\alpha}_\delta(U;E)$ if and only if $\rho^{-\delta}\nabla^j u\in C^{0,\alpha}\left(U;E\otimes T^jM\right)$ for $0\le j\le k$, and the $C^{k,\alpha}_\delta$ norm is equivalent to*

$$\sum_{0\le j\le k}\sup_U\left|\rho^{-\delta}\nabla^j u\right|+\left\|\rho^{-\delta}\nabla^k u\right\|_{0,\alpha;U}$$

Given a Mobius charts $\{\Phi_{p_i}(B_2),\Phi_{p_i}\}_{p_i\in M}$, we will see the transition function and its derivative is uniformly bounded.

Lemma 2.3 *Let (M^{n+1},g) be an asymptotically hyperbolic manifold with boundary regularity $C^{l,\beta}$, $l\ge 2$. Given a Mobius charts covering $\{\Phi_{p_i}(B_2),\Phi_{p_i}\}_{p_i\in M}$, there exists a constant C such that*

$$\|\Phi_{p_j}^{-1}\circ\Phi_{p_i}\|_{C^{l,\beta}(U)}\le C$$

where $U=B_2-\Phi_{p_i}^{-1}(\Phi_{p_i}(B_2)\cap\Phi_{p_j}(B_2))$.

Proof : The transition map can be written down as

$$\begin{aligned}\Phi_{p_j}^{-1}\circ\Phi_{p_i}:\Phi_{p_j}^{-1}(\Phi_{p_i}(B_2)\cap\Phi_{p_j}(B_2))&\to\Phi_{p_j}^{-1}(\Phi_{p_i}(B_2)\cap\Phi_{p_j}(B_2))\\ \mathbf{x}&\mapsto\mathbf{y}\end{aligned}$$

where $\mathbf{x},\mathbf{y}\in B_2\subset\mathbb{H}^{n+1}$. We can thought this as

$$\Phi_{p_j}^{-1}\circ\Phi_{p_i}:\Gamma(U,TM)\to\Gamma(U,TM)$$

Where $\Gamma(U,TM)$ is the section of the tangent bundle on U. Then we have

$$\Phi_{p_j}^{-1}\circ\Phi_{p_i}=\sum_{t=1}^{n+1}\frac{\partial}{\partial x^t}\otimes dx^i\in TM\otimes T^*M$$

Moreover

$$\|\Phi_{p_j}^{-1}\circ\Phi_{p_i}\|=k+1\quad and\quad\nabla\Phi_{p_j}^{-1}\circ\Phi_{p_i}=0$$

By Lemma 2.2, we have

$$\|\Phi_{p_j}^{-1}\circ\Phi_{p_i}\|_{C^{l,\beta}(U)}\le C$$

□

Lemma 2.4 (Lemma 3.5 [Le2006]) *Let (M^{n+1}, g) be an asymptotically hyperbolic manifold with boundary regularity $C^{l,\beta}$, $l \geq 2$. Let u be a global section of a tensor bundle E and $u \in C^{k,\alpha}_{\delta}(M;E)$ with $0 \leq \alpha < 1$ and $0 < k+\alpha \leq l+\beta$. Fix arbitrary $0 \leq \epsilon \leq 2$. Suppose that $\{\Phi_{p_i}(B_2), \Phi_{p_i}\}$ is a Mobius charts covering of M satisfying that*

$$\cup_{p_i} \Phi_{p_i}(B_r) = M \quad \textit{for arbitrary } \epsilon \leq r \leq 2$$

Then we have the following norm equivalence

$$C^{-1} \sup_i \rho(p_i)^{-\delta} \|\Phi_i^* u\|_{k,\alpha;B_r} \leq \|u\|_{k,\alpha,\delta} \leq C \sup_i \rho(p_i)^{-\delta} \|\Phi_i^* u\|_{k,\alpha;B_r}.$$

Proof : Then first inequality is obvious. Because the $\|.\|_{k,\alpha,\delta}$ is defined in the Mobius chart in B_2. For the second inequality, we can make use of Lemma 2.2 to show it. In fact, we only need to show that

$$\|\Phi_{p_i}^* u\|_{C^{k,\alpha}(B_2)} \leq C \sup_j \|\Phi_{p_j}^* u\|_{C^{k,\alpha}(B_r)}$$

Consider all the p_j such that $\Phi_{p_i}(B_2) \cap \Phi_{p_j}(B_r) \neq \emptyset$. Then from lemma 2.2, we have

$$\|\Phi_{p_j}^{-1} \circ \Phi_{p_i}\|_{C^{l,\beta}(U_j)} \leq C$$

where $U_j = B_2 - \Phi_{p_i}^{-1}(\Phi_{p_i}(B_2) \cap \Phi_{p_j}(B_r))$. Then

$$\begin{aligned}
&\|\Phi_{p_i}^* u\|_{C^{k,\alpha}(B_2)} \leq \|\Phi_{p_i}^* u\|_{C^{k,\alpha}(B_r)} + \|\Phi_{p_i}^* u\|_{C^{k,\alpha}(B_2 - B_r)} \\
&\leq \|\Phi_{p_i}^* u\|_{C^{k,\alpha}(B_r)} + \sup_{p_j} \|\Phi_{p_j}^{-1} \circ \Phi_{p_i}\|_{C^{k,\alpha}(U_j)} \times \|\Phi_{p_j}^* u\|_{C^{k,\alpha}(B_r)}
\end{aligned}$$

□

2.3 Interior parabolic estimates

In this section, we will give the parabolic estimate in the weighted space which is just same with the weighted holder space in previous section except there is an extra time dimension. First, we will define the Holder norm and introduce the local parabolic estimate. Then, we will define the weighted Holder norm in asymptotically hyperbolic space and give the proof of the parabolic estimate in this weighted space via Mobius chart. This idea is from [Le2006]. In [Le2006] lemma 4.8, John Lee generalize the elliptic estimates into the weighted space by taking the Mobius chart.

2.3.1 The general parabolic estimate

Most parts of this subsection can be found in chapter 8 of [Kr1996]. In $\mathbb{R}^{n+1+1}$ define the **parabolic distance** between the points $z_1 = (t_1, x_1)$, $z_2 = (t_2, x_2)$ as

$$\rho(z_1, z_2) = |x_1 - x_2| + |t_1 - t_2|^{1/2}$$

Definition 2.6 (Holder Norm) *If u is a function in a domain $Q \subset \mathbb{R}^{n+1+1}$, we denote*

$$[u]_{\alpha,2\alpha;Q} = \sup_{z_1 \neq z_2} \frac{|u(z_1) - u(z_2)|}{\rho^{2\alpha}(z_1, z_2)}, \quad |u|_{\alpha,2\alpha;Q} = |u|_{0:Q} + [u]_{\alpha,2\alpha:Q}$$

where $\alpha \in (0, \frac{1}{2})$

By $C^{\alpha,2\alpha}(Q)$ we denote the space of all functions u for which $|u|_{\alpha,2\alpha;Q} < \infty$. We also introduce the parabolic Holder space $C^{1+\alpha,2+2\alpha}(Q)$ as the set of all real-valued function $u(z)$ defined in Q for which both

$$[u]_{m+\alpha,2m+2\alpha;Q} := \sum_{|l+2k|=2m} \left(\left[D^l \partial_t^k u\right]_{0,0;Q} + \left[D^l \partial_t^k u\right]_{\alpha,2\alpha:Q} \right) < \infty$$

$$|u|_{1+\alpha/2,2+\alpha;Q} := \sum_{|l+2k| \leq 2m} \left(\left[D^l \partial_t^k u\right]_{0,0;Q} + \left[D^l \partial_t^k u\right]_{\alpha,2\alpha:Q} \right) < \infty$$

Let $Q_r(p) = B_r(p) \times [t - r^2, , t]$. Then set

$$|u|'_{m+\alpha,2m+2\alpha;Q_r} = \sum_{|l|+2k \leq 2m} r^{|l|+2k} \left(|D^l \partial_t^k u|_{0,0;Q_r} + r^{2\alpha} \left[D^l \partial_t^k u\right]_{\alpha,2\alpha;Q_r} \right)$$

where ι runs over products of spatial derivatives. Set $B_r = B_r(0) \subset R^n$.

Lemma 2.5 *Let $Q_r = B_r(p) \times [t - r^2, t]$ and $Q_{2r} = B_{2r}(p) \times [t - 4r^2, t]$ and $u \in C^{1+\alpha,2+2\alpha}(Q_{2r})$, $\alpha \in (0, \frac{1}{2})$ satisfying that $(\partial_t - L)u = f$ where*

$$Lu = a_{ij}(x)\partial_{ij}^2 u + b_i(x)\partial_i u + c(x)u$$

such that $\frac{1}{\Lambda} < a_{ij} < \Lambda$, $|a_{ij}|'_{m-1+\alpha,2m-2+2\alpha;Q_{2r}} < \Lambda$, $|b_i|'_{m-1+\alpha,2m-2+2\alpha;Q_{2r}} < r^{-1}\Lambda$ and $|c(x)|'_{m-1+\alpha,2m-2+2\alpha;Q_{2r}} < r^{-2}\Lambda$. Then, we have

$$|u|'_{m+\alpha,2m+2\alpha;Q_r} \leq C(r^2|f|'_{m-1+\alpha,2m-2+2\alpha;Q_{2r}} + |u|_{0,0;Q_{2r}})$$

where C depends only on Λ, m, α, n

Proof For m = 1, the Lemma is exactly the same as Theorem 8.11.1 in [Kr1996] and for $m > 1$ it follows by differentiation. □

We notice that in the Lemma 2.5 we need to require that $t - r^2 \geq 0$ which implies that $t - r^2 > 0$. Therefore, the parabolic neighborhood Q_r can not be taken from the initial time. In order to make the local estimate can be taken the initial time, we need to do an extension of the solution $u(t, x)$ from $t \in [0, T)$ to $t \in (-C, T)$ where $C > 0$. By this way, if we take $t = r^2$, then $Q_r(p, r^2) = B_r(p) \times [0, r^2]$ and $Q_{2r}(p, r^2) = B_{2r}(p) \times [-3r^2, r^2]$. Then apply the above theorem at $t = 0$.

Next, we are going to generalized lemma 2.5 into the weighted space on asymptotically hyperbolic space. Let $\bar{M}$ be a connected smooth $(n{+}1)$-manifold, g is a metric on M that is asymptotically hyperbolic of class $C^{l,\beta}$, with $l \geq 2$ and $0 \leq \beta \leq 1$ and ρ is a fixed smooth defining function for ∂M. Let E be a geometric tensor bundle on $M \times \mathbb{R}$. Let $(\Phi_\alpha(B_2), \Phi_\alpha)$ be the Mobius covering of M. For any tensor field u with locally $C^{m+\alpha,2m+2\alpha}$ coefficients, define the norm $|u|_{m+\alpha,2m+2\alpha}$ by

$$|u|_{m+\alpha,2m+2\alpha} := \sup_{\Phi} |\Phi^* u|_{C^{m+\alpha,2m+2\alpha}(B_2\times\mathbb{R})}$$

where $|\Phi^* u|_{m+\alpha,2m+2\alpha;B_2\times\mathbb{R}}$ is just the usual Euclidean Holder norm of the components of u on $B_2 \subset \mathbb{H}$, and the supremum is over all Mobius charts defined on B_2. Let $C^{m+\alpha,2m+2\alpha}(M \times \mathbb{R}, E)$ be the space of sections of E for which this norm is finite.

The **weighted Holder space** on $\mathbb{R}^{n+1} \times \mathbb{R}$ is defined as

$$\begin{aligned} C^{m+\alpha,2m+2\alpha}_{\delta}(M \times \mathbb{R}; E) :=& \rho^\delta C^{m+\alpha,2m+2\alpha}(M \times \mathbb{R}; E) \\ =& \left\{ \rho^\delta u : u \in C^{m+\alpha,2m+2\alpha}(M \times \mathbb{R}; E) \right\} \end{aligned}$$

with norm

$$|u|_{m+\alpha,2m+2\alpha,\delta} := |\rho^{-\delta} u|_{m+\alpha,2m+2\alpha}$$

Remark 2.3 *If $U \subset M \times \mathbb{R}$ is a subset, the restricted norms are denoted by $|\cdot|_{m+\alpha,2m+2\alpha,\delta;U}$ and the space $C^{m+\alpha,2m+2\alpha}_{\delta}(U, E)$ are the spaces of section over U for which these norms are finite.*

2.3.2 The parabolic estimate on weighted space

Lemma 2.6 *Let (M^{n+1}, g_+) be an asymptotically hyperbolic manifold with $C^{2,\alpha}$ regularity. For the heat equation*

$$\frac{\partial}{\partial t} u = \Delta_{L(g_+)} u + f$$

where u is a smooth section of symmetric two tensor on M and is also a solution of the above equation for $t \in (\eta, T]$. Then, we have the following estimate

$$|u|_{m+\alpha,2m+2\alpha,\delta;M\times(\eta,T]} \leq C(|f|_{m-1+\alpha,2m-2+2\alpha,\delta;M\times(\eta,T]} + |u|_{0,0,\delta;M\times(\eta,T]})$$

where $C = C(g_+, n, m, \alpha, \delta)$.

Moreover, if there is another asymptotically hyperbolic metric g satisfying that

$$|g - g_+|_{2,0,M} \leq \epsilon$$

Then for the equation

$$\frac{\partial}{\partial t}u = \Delta_{L(g)}u + f,$$

its solution u also satisfies the following parabolic estimate

$$|u|_{m+\alpha,2m+2\alpha,\delta;M\times(\eta,T]} \le C(|f|_{m-1+\alpha,2m-2+2\alpha,\delta;M\times(\eta,T]} + |u|_{0,0,\delta;M\times(\eta,T]})$$

where $C = C(g_+, \epsilon, n, m, \alpha, \delta, \eta, T)$.

Proof: Take an arbitrary defining function $\rho \in C^\infty(\bar{M})$ for the asymptotically hyperbolic metric g_+. We can choose a covering of a neighborhood of ∂M in $\bar{M}$ by finitely many smooth coordinate charts (Ω, Θ), where each coordinate map Θ is of the form $\Theta = (\theta, \rho) = \left(\theta^1, \ldots, \theta^n, \rho\right)$ and extends to a neighborhood of $\bar{\Omega}$ in $\bar{M}$.(See section 2.2) Then take a Mobius charts covering of M based on the above background coordinate

$$\{\Phi_{p_i}(B_2), \Phi_{p_i}\}$$

where

$$B_r = \{(x,y) \in \mathbb{H} : d_{\breve{g}}((x,y),(0,1)) < r\} \quad x = (x^1, \cdots, x^n)$$

and

$$(\theta, \rho) = \Phi_{p_0}(x,y) = (\theta_0 + \rho_0 x, \rho_0 y)\,.$$

And

$$\mathbb{H} = \mathbb{H}^{n+1} \triangleq \left\{\left(x^1, \cdots, x^n, y\right) \subset \mathbb{R}^{n+1} : y > 0\right\}$$

endowed with the hyperbolic metric $\breve{g} = y^{-2}\sum_i \left(dx^i\right)^2$. Since

$$\cosh(d_{\breve{g}}(x,y),(0,1))) = \frac{|x|^2 + (y-1)^2}{2y} + 1$$

where $|x|^2 = \sum_{i=1}^n (x^i)^2$ (See detail in [JR2006]). Therefore,

$$B_r = \{(x,y) \in \mathbb{H} : |x|^2 + (y - \cosh(r))^2 \le \cosh^2(r) - 1\}$$

Let

$$\mathbb{B}_a(x_0, y_0) \triangleq \{(x, y \in \mathbb{H}) : |x - x_0|^2 + (y - y_0)^2 \le a^2\}$$

where $x_0 = (x_0^1, \cdots, x_0^n)$ and $|x - x_0|^2 = \sum_{i=1}^n (x^i - x_0^i)^2$. Therefore,

$$\mathbb{B}_a(0,1) \subset B_2 \quad if \quad 0 \le a \le \sqrt{\cosh(2) - 1}(\sqrt{\cosh(2)+1} - \sqrt{\cosh(2)-1})$$

And

$$B_r \subset \mathbb{B}_a(0,1) \quad if \quad \cosh(r) \le \frac{(1+a)^2 + 1}{2(1+a)}$$

Then, take $a = \sqrt{\cosh(2) - 1}(\sqrt{\cosh(2)+1} - \sqrt{\cosh(2)-1})$. Then, we have two parabolic ball in B_2

$$Q_a(0,1) \triangleq \mathbb{B}_a(0,1) \times (t - a^2, t] \quad and \quad Q_{\frac{a}{2}}(0,1) \triangleq \mathbb{B}_{\frac{a}{2}}(0,1) \times (t - \frac{a^2}{4}, t]$$

Moreover, by

$$|g - g_+|_{2,0;M} \leq \epsilon,$$

we have the coefficients of $\Delta_L(g)$ is uniformly bounded and satisfies the condition of lemma 2.5. Then we can use the lemma 2.5 in each $Q_a(0,1)$ and $Q_{\frac{a}{2}}(0,1)$. Then, by Lemma 2.4, we have the result. □

Corollary 2.1 *Let (M^{n+1}, g_+) be an asymptotically hyperbolic manifold with regularity $C^{2,\alpha}$. Then, for corresponding curvature evolution flow of the normalized Ricci flow*

$$\frac{\partial}{\partial t} h_{il} = \Delta_L h_{il} - 2n h_{il},$$

we have

$$|h|_{m+\alpha, 2m+2\alpha, \delta; M\times(\eta,T]} \leq C(|h|_{0,0,\delta;M\times(\eta,T]})$$

where $C = C(g_+, \epsilon, n, m, \alpha, \delta)$.

2.4 Short time existence of the curvature evolution flow on weighted space

For the short time existence of the Ricci flow on complete Riemannian manifold, we will quote the result of [Shi1989] [Mi2002]

Theorem 2.1 (Short time existence, Theorem 1.1 [Shi1989]) *Let $(M, g_{ij}(x))$ be an n -dimensional complete noncompact Riemannian manifold with its Riemannian curvature tensor $\{R_{ijkl}\}$ satisfying*

$$|R_{ijkl}|^2 \leq k_0 \quad \text{on } M$$

where $0 < k_0 < +\infty$ is a constant. Then there exists a constant $T(n, k_0) > 0$ depending only on n and k_0 such that the evolution equation

$$\tfrac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t) \text{ on } M$$

$$g_{ij}(x,0) = g_{ij}(x) \quad \forall x \in M$$

has a smooth solution $g_{ij}(x,t) > 0$ for a short time $0 \leq t \leq T(n,k_0)$, and satisfies the following estimates: For any integer $m \geq 0$, there exist constants $C_m > 0$ depending only on n, m and k_0 such that

$$\sup_{x\in M} |\nabla^m R_{ijkl}(x,t)|^2 \leq C_m / t^m, \quad 0 \leq t \leq T(n,k_0)$$

From Theorem 2.1, we have the short time existence of the solution for the evolution equation for the normalized Ricci flow.

Corollary 2.2 *For the curvature evolution equation*

$$\frac{\partial}{\partial t} h_{il} = \Delta_L h_{il} - 2n h_{il}$$

where $h(t) = Ric_{g(t)} + ng(t)$*. If*

$$\sup_{x\in M} |h|(x,0) \le \epsilon_0 \quad and \quad \sup_{x\in M} |\nabla h|(x,0) \le \epsilon_1,$$

then there exists a C_0 *and* C_1 *such that the solution* $h(x,t)$ *obtained form Theorem 2.1 satisfy that*

$$\sup_{x\in M} \|h\|(x,t) \le C_0\epsilon_0 \quad and \quad \sup_{x\in M} \|\nabla h\|(x,t) \le C_1\epsilon_1$$

Proof : First, we have the following evolution equation

$$\frac{\partial}{\partial t}\|h\|^2 \le \Delta\|h\|^2 - 2\|\nabla h\|^2 + C\|h\|^2$$

Therefore, we have

$$\frac{\partial}{\partial t}\|h\|^2 \le \Delta\|h\|^2 + C\|h\|^2$$

The following is the from the [LY2010], we first consider the cut-off function

$$\xi(y,s) = -\frac{d_0^2(y)}{(2 + C_2\varepsilon)(t-s)}$$

where $d_0(y)$ is the distance function from the point y to the geodesic ball $B_0(x, \sqrt{\tau})$ with respect to the initial metric g_0 and C_2 is chosen so that

$$\xi_s + \frac{1}{2}\|\nabla\xi\|^2 \le 0$$

We then set

$$J(s) = \int_{M^n} \|h\|^2(y,s)e^{\xi(y,s)}\mathrm{d}y$$

Because the curvature is bounded. By the Bishop-Gromov volume comparison theorem, the volume is at worst exponential blow up. With this volume growth condition and the fact that $g(t)$ are all quasi-isometric to g_0 one sees that $J(s)$ is finite for all $0 < s < t < T$. The important observation here is that the non-degeneracy implies the exponential decay of $J(s)$ when it evolves. We compute

$$\frac{\mathrm{d}J}{\mathrm{d}s}(s) = \int_M 2\left\langle \Delta h_{ij} - 2R_{ipjq}h_{pq} - 2h_{ip}h_{pj}, h_{ij}\right\rangle \mathrm{e}^{\xi}$$

$$+ \|h\|^2\mathrm{e}^{\xi}\xi_s\mathrm{d}y + C\varepsilon J(s)$$

$$\int_M 2(\Delta h)h\mathrm{e}^{\xi} + \|h\|^2\mathrm{e}^{\xi}\xi_s\mathrm{d}y \le \int_M 2\left\langle \Delta\left(\mathrm{e}^{\frac{\xi}{2}}h\right), \left(\mathrm{e}^{\frac{\xi}{2}}h\right)\right\rangle$$

and therefore,

$$\frac{\mathrm{d}J}{\mathrm{d}s}(s) \le -2\int_M < (\Delta_L + 2(n-1))\left(\mathrm{e}^{\frac{\xi}{2}}h\right), \left(\mathrm{e}^{\frac{\xi}{2}}h\right) > +C\varepsilon J(s)$$

Since there always exists a lower bounded for the L^2 spectrum of Δ_L, then we have

$$J(s) \leq \mathrm{e}^{\lambda s} J(0)$$

where λ is the lower bounded of the spactrum of $\Delta_L + 2(n-1)Id$. By Morser iteration, we have

$$\sup_{B_0\left(x,\sqrt{\frac{\tau}{2}}\right)\times\left[t-\frac{\tau}{2},t\right]} \|h\|^2 \leq C\left(n,\tau,k_0\right)\int_{t-\tau}^{t}\int_{B_0(x,\sqrt{\tau})} \|h\|^2(y,s)\mathrm{d}y\mathrm{d}s$$

Then, we have

$$\sup_{B_0\left(x,\sqrt{\frac{\tau}{2}}\right)\times\left[t-\frac{\tau}{2},t\right]} \|h\|^2 \leq C\mathrm{e}^{\lambda t}\int_M \|h\|^2 \mathrm{e}^{\frac{d_0^2}{(2+C\varepsilon)t}}\mathrm{dv}_{\mathrm{g}(0)}$$

Then

$$\sup_{B_0\left(x,\sqrt{\frac{\tau}{2}}\right)\times\left[t-\frac{\tau}{2},t\right]} \|h\|^2 \leq C\mathrm{e}^{\lambda+\lambda_0 t}\int_M \|h\|^2 \mathrm{e}^{-\frac{d_0^2}{(2+C\varepsilon)t}-\lambda_0 t}\mathrm{dv}_{\mathrm{g}(0)}$$

We have

$$-\frac{d_0^2}{(2+C\varepsilon)t} \leq -2\sqrt{\frac{\lambda_0}{(2+C\epsilon)}}d_0.$$

We only need to choose enough large λ_0 such that

$$\int_M e^{-2\sqrt{\frac{\lambda_0}{(2+C\epsilon)}}d_0} dV_{g(0)} \leq \infty$$

Then, we can get

$$\sup_{x\in M}\|h\|(x,t) \leq C_0\epsilon_0$$

Then, by the standard parabolic estimate, we have

$$\sup_{x\in M}\|\nabla h\|(x,t) \leq C_1\epsilon_1$$

□

Therefore, we have the short time existence of the curvature evolution flow. Here, we only need to show that the solution of the curvature evolution is in the weighted space. In order to show this, we need the following maximal principal.

Lemma 2.7 (Lemma 4.2 [QSW2013]) *Suppose that $(M^{n+1}, g(t))$ is a smooth family of complete Riemannian manifolds with boundary ∂M for $t \in [0,T]$. Let u be a function on $M \times [0,T]$ which is smooth on $M \times (0,T]$ and continuous on $M \times [0,T]$. Assume that u and $g(t)$ satisfy* *(i) the differential inequality*

$$\frac{\partial}{\partial t}u - \Delta_{g_t} u \leq \boldsymbol{a} \cdot \nabla u + bu$$

where the vector a and the function b are uniformly bounded

$$\sup_{M\times[0,T]} |\boldsymbol{a}| \leq \alpha_1, \qquad \sup_{M\times[0,T]} |b| \leq \alpha_2$$

with some constants $\alpha_1, \alpha_2 < \infty$

(ii)

$$\sup_M u(x,0) \leq 0$$

and

$$\sup_{\partial M\times[0,T]} u(x,t) \leq 0$$

(iii)

$$\int_0^T \int_M \exp\left[-\alpha_3 d^t(y,p)^2\right] u_+^2(y) dv_t(y) dt < \infty$$

for some positive number α_3.

(iv)

$$\sup_{M\times[0,T]} \left|\frac{\partial}{\partial t} g(x,t)\right| \leq \alpha_4$$

with some constant $\alpha_4 < \infty$.

Then we have $u \leq 0$ on $M \times [0,T]$.

Theorem 2.2 (Lemma 4.3 [QSW2013]) *Suppose that $g(t), t \in [0,T]$, is a solution of normalized Ricci flow starting from an asymptotically hyperbolic metric g_+ satisfying $\|Rm\|_{L^\infty} \leq k_0$, $\|\nabla Rm\|_{L^\infty(M)} \leq k_1$. Then there exist numbers C, depending on k_0, k_1, n, C_0, and T such that*

$$|h|_{0,0,\delta;M} \leq CC_0, \quad |h|_{1,0,\delta;M} \leq CC_0 \quad \textit{and} \quad [h]_{2,0,\delta;M} \leq \frac{CC_0}{\sqrt{t}}$$

for all $t \in [0,T]$, if

$$|h|_{1,0,\delta;M} \leq C_0$$

Proof : See Lemma 4.3 in [QSW2013]. From the appendix, for the curvature flow

$$\frac{\partial}{\partial t} h_{il} = \Delta_L h_{il} - 2n h_{il}$$

where $h(t) = Ric_{g(t)} + ng(t)$. Then we can get the evolution equation for the L^2 norm of h

$$\frac{\partial}{\partial t}\|h\|^2 \leq \Delta\|h\|^2 - 2\|\nabla h\|^2 + C\|h\|^2$$

$$\frac{\partial}{\partial t}\|\nabla h\|^2 \le \Delta\|\nabla h\|^2 - 2\left\|\nabla^2 h\right\|^2 + C\left(\|h\|^2 + \|\nabla h\|^2\right)$$

$$\begin{aligned}\frac{\partial}{\partial t}\left(t\left\|\nabla^2 h\right\|^2\right) \le& \Delta\left(t\left\|\nabla^2 h\right\|^2\right) - 2t\left\|\nabla^3 h\right\|^2 \\ &+ (1+Ct)\left\|\nabla^2 h\right\|^2 + C\left(\|h\|^2 + \|\nabla h\|^2\right)\end{aligned}$$

Let ρ be a fixed geodesic defining function of the asymptotically hyperbolic metric g_0, one knows the fact that $|\Delta_g \rho| \le C\rho$ and $\|\nabla_g \rho\|^2 \le C\rho^2$. To estimate $\left|\Delta_{g(t)}\rho\right|$ and $\left\|\nabla_{g(t)}\rho\right\|^2_{g(t)}$ we recall again

$$\frac{\partial \Gamma^k_{ij}}{\partial t} = -g^{kl}\left(R_{li,j} + R_{lj,i} - R_{ij,l}\right)$$

and thus calculate

$$\begin{aligned}\frac{\partial}{\partial t}(\Delta r) &= \frac{\partial}{\partial t}\left(g^{ij}\left(\nabla^2 r\right)_{ij}\right) \\ &= 2g^{ki}g^{lj}h_{kl}\left(\nabla^2 r\right)_{ij} + g^{ij}g^{kl}\left(R_{li,j} + R_{lj,i} - R_{ij,l}\right)\nabla_k r \\ &= 2g^{ki}g^{lj}h_{kl}\left(\nabla^2 r\right)_{ij}\end{aligned}$$

Form the fact that $C^{-1}g \le g(t) \le Cg$ and the property of the asymptotically hyperbolic spaces, we get the estimates

$$\left|\Delta_{g(t)}\rho\right| \le C\rho \quad \text{and} \quad \left\|\nabla_{g(t)} r\right\|^2_{g(t)} \le Cr^2$$

We consider $\bar{h} = \rho^{-\gamma}h, \bar{\nabla}h = \rho^{-\gamma}\nabla h, \bar{\nabla^2}h = \rho^{-\gamma}\nabla^2 h$, and $\bar{\nabla^3}h = \rho^{-\gamma}\nabla^3 h$ and calculate

$$\frac{\partial}{\partial t}\|\bar{h}\|^2 \le \Delta\|\bar{h}\|^2 - \|\bar{\nabla}h\|^2 + C\|\bar{h}\|^2$$

$$\frac{\partial}{\partial t}\|\bar{\nabla}h\|^2 \le \Delta\|\bar{\nabla}h\|^2 - \left\|\bar{\nabla^2}h\right\|^2 + C\left(\|\bar{h}\|^2 + \|\bar{\nabla}h\|^2\right)$$

$$\begin{aligned}\frac{\partial}{\partial t}\left(t\left\|\bar{\nabla^2}h\right\|^2\right) \le& \Delta\left(t\left\|\bar{\nabla^2}h\right\|^2\right) - t\left\|\bar{\nabla^3}h\right\|^2 \\ &+ (1+Ct)\left\|\bar{\nabla^2}h\right\|^2 + C\left(\|\bar{h}\|^2 + \|\bar{\nabla}h\|^2\right)\end{aligned}$$

Set

$$\varphi_1 = \|\bar{h}\|^2 + \|\bar{\nabla}h\|^2$$

and

$$\varphi_2 = \|\bar{h}\|^2 + \|\bar{\nabla}h\|^2 + t\left\|\bar{\nabla^2}h\right\|^2$$

and calculate that

$$\frac{\partial}{\partial t}\varphi_1 \le \Delta\varphi_1 + C\varphi_1$$

and

$$\frac{\partial}{\partial t}\varphi_2 \le \Delta\varphi_2 + C\varphi_2$$

Therefore

$$\frac{\partial}{\partial t}\left(\mathrm{e}^{-Ct}\varphi_1\right) \leq \left(\mathrm{e}^{-Ct}\Delta\varphi_1\right)$$

$$\frac{\partial}{\partial t}\left(\mathrm{e}^{-Ct}\varphi_2\right) \leq \left(\mathrm{e}^{-Ct}\Delta\varphi_2\right)$$

By Lemma 2.7, we can have the result.

□

Corollary 2.3 *Under the assumption of the above theorem, there also exists a number C_1, depending on k_0, k_1, n, C_0 and T such that*

$$|g(t) - g(0)|_{2,0,\delta;M} < C_1$$

Proof : By the fact that $g(t) = \int_0^t h(\tau, .)d\tau$, we can easily get the result form theorem 2.2. □

2.5 Semigroup and its generators

In this section, we will introduce some basic concept of the semigroup and its generator. For more detail, refers to [Ev2010] 7.4

Definition 2.7 (Semigroup) *$S(t)$ is called the semi-group if it satisfies that*

- *$\{S(t)\}_{t\geq 0}$ is a family of bounded linear mapping from the Banach space X to X*
- *$S(0) = Id_X$*
- *$S(t+s) = S(t)S(s) = S(s)S(t)$*
- *$t \mapsto S(t)u$ is continuous from $[0, \infty)$ to X*

Definition 2.8 (Generator of semigroup) *Write*

$$D(A) := \left\{ u \in X | \lim_{t\to 0+} \frac{S(t)u - u}{t} \text{ exists in } X \right\}$$

and

$$Au := \lim_{t\to 0+} \frac{S(t)u - u}{t} \quad (u \in D(A))$$

We call $A : D(A) \to X$ the (infinitesimal) generator of the semigroup $\{S(t)\}_{t\geq 0}$; $D(A)$ is the domain of A.

There are some basic properties about the semigroup and its generator.

Theorem 2.3 *Assume $u \in D(A)$. Then*

1) $S(t)u \in D(A)$ for each $t \geq 0$.

2) $AS(t)u = S(t)Au$ for each $t \geq 0$.

3) The mapping $t \mapsto S(t)u$ is differentiable for each $t > 0$.

4) $\frac{d}{dt}S(t)u = AS(t)u$ $(t > 0)$.

Proof : See 7.4.1 Theorem 1 in [Ev2010]. □

Definition 2.9 (Resolvent set) *We say a real number λ belongs to $\rho(A)$, the resolvent set of A, provided the operator*

$$\lambda I - A :\to X$$

is on to one and onto. And if $\lambda \in \rho(A)$, the resolvent operator $R_\lambda : X \to X$ is defined by $R_\lambda u := (\lambda I - A)^{-1}u$

Remark 2.4 *According to the Closed Graph Theorem, $R_\lambda : X \to D(A) \subset X$ is bounded linear operator.*

Theorem 2.4 (Hille-Yosida-Phillips) *Let A be a closed, densely defined linear operator on X. Then A is the generator of a semigroup $\{S(t)\}_{t\geq 0}$ if and only if*

$$(c, \infty) \subset \rho(A) \quad \text{and} \quad \|R_\lambda\| \leq \frac{1}{\lambda - c} \text{ for } \lambda > c$$

Moreover, we have $||S(t)|| \leq e^{-ct}$

Proof : See 7.4.2 Theorem 4 in [Ev2010]. □

Now, let (M^{n+1}, g_+) be an asymptotically hyperbolic manifold and take $X = C^{0,\alpha}_\delta(Sym^2T^*M^{n+1})$ with $\delta \in (0, n)$ and trivial L^2 kernel of P on $Sym^2T^*M^{n+1}$. By the lemma 3.7 of [Le2006], the $P = \Delta_L + 2nId$ is an isomorphism from $C^{2,\alpha}_\delta$ to $C^{0,\alpha}_\delta$. Then we have

$$||Pu||_{C^{0,\alpha}_\delta} \geq c||u||_{C^{0,\alpha}_\delta}$$

where $c > 0$. And for $c \geq -\lambda$, we have

$$||Pu + \lambda u||_{C^{0,\alpha}_\delta} \geq (\lambda + c)||u||_{C^{0,\alpha}_\delta}$$

Therefore,

$$(-c, \infty) \subset \rho(A) \quad \text{and} \quad \|R_\lambda\| \leq \frac{1}{\lambda + c} \text{ for } \lambda > -c$$

Therefore, P is a generator of a semigroup $S(t)$ with $|S(t)| \leq e^{-ct}$

3 Long time existence

3.1 The history

The long time existence, is more complicated. For the closed manifold, we have the result of [Ye1993] which just tell us if the pinching curvature is small enough, then we have the long time existence and convergence of the normalized Ricci flow. For the noncompact complete manifold, we have results of [LY2010], [QSW2013], [SSS2010], [Ba2015] which, roughly speaking, just tell us if the smallest L^2 eigenvalue of the Lichnerowicz operator is positive, then we have the long time existence and convergence of the Ricci flow. (Roughly speaking, this is because that the positive smallest eigenvalue just implies the exponential decay of the semigroup with respect to time [Hille-Yosida-Phllips]) Therefore, in order to show the long time existence, the key is to get a precise eigenvalue estimate of the Lichnerowicz operator. Generally, we have the following theorem,

Lemma 3.1 *If for any smooth compact support function u, we have*

$$(u, \Delta u) \geq \lambda u$$

with some constant λ. Then, for any smooth compact support tensor field ω, we have

$$(\omega, \Delta\omega) \geq \lambda\omega$$

This theorem just tell us that once we have eigenvalue estimate for function, we will have a rough eigenvalue estimate for tensor. But this is not so precise. Actually, by the method [Bi1999] [Le2006] [Ba2015], for the symmetric spaces, the smallest eigenvalue for tensor is always strictly bigger than the eigenvalue for function (The difference of this two eigenvalue is the eigenvalue of Casimir operator. See I.2 in [Bi1999]). Then, by parametrix method of Proposition 6.2 in [Le2006] and Proposition I.3.5 in [Bi1999], we have the isomorphism theorem [Lemma 7.5 Le].

In [QSW], the reason that they did not fully recover the Lemma A in [Le] by Ricci flow, is because that they just use the smallest eigenvalue for function to estimate the smallest eigenvalue for symmetric two tensor. By this reason, they need to require stronger nondegeneracy. See [Theorem 1.4 QSW]. In our method, we just make use of the [theorem 7.5 Le] to get the exponential decay of the semigroup and then by the argument of [Ba2015], we can fully recover the Lemma A in [Le].

3.2 The main lemma

Let (M^{n+1}, g_+) be an asymptotically hyperbolic space with nondegeneracy $\lambda > 0$ and regularity $C^{2,\alpha}$. From the section 2.5 and let $S(t) : C^{0,\alpha}_\delta \to C^{0,\alpha}_\delta$ be the semi group for the linear operator $\Delta_{L(g(t-l))} : C^{0,\alpha}_\delta \to C^{0,\alpha}_\delta$. Then, we have the solution of the equation of

$$\frac{\partial}{\partial t} h_{il} = \Delta_{L(g(t-l))} h_{il} - 2n h_{il} + Q$$

can be written as

$$h(t,x) = S(t)(h(t-l,x)) + \int_l^{t-l} S(t-\tau)Q(\tau,x))d\tau$$

Remark 3.1 *Form the Hill-Yosida-Phillips theorem, for small enough $\epsilon > 0$, if $|g^{k-1}(t) - g_+|_{C^1} \le \epsilon$ and $\delta \in (0, n-1)$, then there exists $0 < \lambda_\epsilon < \lambda$ such that $|S(t)|_{C^{0,\alpha}_\delta} \le C\exp(-\lambda_\epsilon t)$.*

Theorem 3.1 *Let (M^{n+1}, g_+) be an asymptotically hyperbolic manifolds with nondegeneracy $\lambda > 0$, regularity $C^{2,\alpha}$ and $\|\nabla Rm\|_{L^\infty} \le k_1$. Then, for any $\delta \in (0, n)$, there exists $\epsilon_0(\lambda, k_1) > 0$ such that if $|h|_{0,0,\delta;M} \le \epsilon_0$, the solution of the normalized Ricci flow $g(t,x)$ has long time existence and $g(t,x)$ converges to an Einstein manifold in the sense of C^2_δ norm. Moreover, the limit metric is an Asymptotically hyperbolic Einstein metric with the same conformal infinity.*

Proof of Theorem 3.1

Step 1 : (Finding the iterating inequality) The following $|.|$ refers to the $C^{0,\alpha}_\delta$. Given a $\epsilon > 0$. Let $T_{max}(\epsilon)$ be the supreme of T such that for arbitrary $t \in [0, T_{max}]$, $|\nabla^m_{g(t)}(g(t)) - g(0))|_{0,0,\delta:M} \le \epsilon$, where $m = 0, 1, 2$. Then, for the equation,

$$\frac{\partial}{\partial t} h_{il} = \Delta_{L(g(t-l))} h_{il} - 2n h_{il} + Q$$

we have

$$|Q|_\le C(\epsilon)\epsilon|h|$$

Then,

$$|h(t,x)| \le e^{-\lambda_\epsilon l}|h(t-l,x)| + C(\epsilon)\epsilon\int_{t-l}^t e^{-\lambda_\epsilon(t-\tau)}|h(\tau,x)|d\tau$$

let $G(t,x) = \underset{\tau\in[t-l,t]}{max} e^{\lambda_\epsilon\tau}|h^K(\tau,x)|$. Then, we have

$$G(t,x) \le |h(t-l,x)| + C(\epsilon)\epsilon l G(t,x)$$

Therefore,

$$G^k(t,x) \le \frac{|h^K(t-l,x)|}{(1 - C_1\epsilon l)}$$

which implies that

$$|h(t,x)| \le \frac{e^{-\lambda_\epsilon l}}{(1 - C_1\epsilon l)}|h(t-l,x)|$$

Step 2 : (Determine the ϵ and l) Consider the function $f(l) = e^{-\lambda_\epsilon l} + C_1\epsilon l - 1$. Then, $f(0) = 0$ and $f'(0) = -\lambda_\epsilon + C_1\epsilon$. We see that as $\epsilon \to 0$, $\lambda_\epsilon \to \lambda$. Therefore, we can take a proper ϵ such that $f'(0) < 0$. We can also find a positive number $L_1(\epsilon) > 0$ such that for arbitrary $0 < l < L_1$, we have $f'(l) \leq 0$ and $f'(L_1) = 0$.

Denote

$$q(s, \epsilon) \stackrel{d}{=} \text{Max}_{s \leq l \leq min\{T_{max}(\epsilon), L_1(\epsilon)\}} \frac{e^{\lambda_\epsilon}}{1 - C_1\epsilon l}$$

where s is a small positive to be determined. We see that $q(s, \epsilon) < 1$ and as $s \to 0$, $q(s, \epsilon) \to 1$. Therefore, for arbitrary $s \leq l \leq \min\{T_{max}(\epsilon), L_1(\epsilon)\}$,

$$|h(t, x)| \leq \frac{1}{q(s, \epsilon)} |h(t - l, x)|$$

Now, fix an above ϵ and so the $L_1(\epsilon)$ and $T_{max}(\epsilon)$ are also fixed. There always exists a large enough integer N such that $\frac{T_{max}(\epsilon)}{N} < L_1$

Again, fix a such N. Take $s = \frac{T_{max}(\epsilon)}{N+1}$. Then, we have $q(s, \epsilon) < 1$. And for $s = \frac{T_{max}(\epsilon)}{N+1} \leq l \leq \min\{T_{max}(\epsilon), L_1\}$,

$$|h(t, x)| \leq \frac{1}{q(s, \epsilon)} |h(t - l, x)|$$

Then, by corollary 2.1, we also have

$$|\nabla^m h(t, x)| \leq \frac{C(m, \epsilon)}{q(s, \epsilon)} |h(t - l, x)|$$

Moreover, by theorem 2.2, for $t \in [0, s]$, we have

$$|h| \leq C(s)\epsilon_0, \quad |\nabla h| \leq C(s)\epsilon_0 \quad and \quad |\nabla^2 h| \leq \frac{C(s)\epsilon_0}{\sqrt{t}}$$

Then, for $t \in [ks, (k+1)s]$ where k is an integer, we have

$$|h| \leq \frac{C(s)}{q(s, \epsilon)^k} \epsilon_0, \quad |\nabla h| \leq \frac{C(s)}{q(s, \epsilon)^k} \epsilon_0 \quad |\nabla^2 h| \leq \frac{C(s)\epsilon_0}{q(s, \epsilon)^k \sqrt{t}}$$

Then, for arbitrary $t > T_{max}(\epsilon)$, we can always take an integer K and $l \in [\frac{T_{max}}{N+1}, \frac{T_{max}(\epsilon)}{N}]$ such that $T_{max}(\epsilon) < t = Kl < (K+1)l$ and

$$|g(Kl, x) - g(0, x)| \leq \sum_{k=1}^{K} \int_{(k-1)s}^{ks} |h(\tau, x)| d\tau \leq \frac{C(s)s}{1 - q(s, \epsilon)} \epsilon_0$$

$$|\nabla(g(Kl, x) - g(0, x))| \leq \sum_{k=1}^{K} \int_{(k-1)s}^{ks} |\nabla h(\tau, x)| d\tau \leq \frac{C(s)s}{1 - q(s, \epsilon)} \epsilon_0$$

$$|\nabla^2(g(Kl,x)-g(0,x))| \leq \sum_{k=1}^{K} \int_{(k-1)s}^{ks} |\nabla^2 h(\tau,x)| d\tau \leq \sum_{k=1}^{K} \int_{(k-1)s}^{ks} \frac{C(s)\epsilon_0}{q(s,\epsilon)^k \sqrt{\tau}} d\tau \leq \frac{C(s)\sqrt{s}}{1-q(s,\epsilon)} \epsilon_0$$

we can always take $\epsilon_0 > 0$ small enough such that

$$\frac{s}{1-q(s,\epsilon)}\epsilon_0 \leq \frac{\epsilon}{2} \quad \frac{C(s)s}{1-q(s,\epsilon)}\epsilon_0 \leq \frac{\epsilon}{2} \quad \frac{C(s)s}{1-q(s,\epsilon)}\epsilon_0 \leq \frac{\epsilon}{2}$$

which is contradict to the choice of T_{max}. Therefore, we have long time existence.

Step 3: (Convergence) Under the assumption of step 2. We will show that for arbitrary ϵ_1, there exists a $T(\epsilon_1)$ such that as long as the $t_1, t_2 \geq T(\epsilon_1)$, we have

$$|\nabla^m (g(t_1,x) - g(t_2,x))| \leq \epsilon_1$$

for $m = 0, 1, 2$.

For arbitrary $T > 0$, there exists an unique posititve integer k, such that $0 \leq T - ks < s$. By the previous discussion, we have

$$|h(T,x)| \leq \frac{1}{q(s,\epsilon)^k} |h(T-ks,x)|$$

By theorem 2.2, for $t \in [0,s]$, we have

$$|h| \leq C(s)\epsilon_0, \quad |\nabla h| \leq C(s)\epsilon_0 \quad and \quad |\nabla^2 h| \leq \frac{C(s)\epsilon_0}{\sqrt{t}}$$

Therefore,

$$|h(T,x)| \leq \frac{1}{q(s,\epsilon)^k} C(s)\epsilon_0$$

There exists an integer $K(\epsilon_1)$ such that if $k > K(\epsilon_1)$, we have

$$\frac{1}{q(s,\epsilon)^k} C(s)\epsilon_0 \leq \epsilon_1$$

Therefore, as long as $T \geq (K(\epsilon_1)+1)s$, we have

$$|h(T,x)| \leq \epsilon_1$$

For arbitrary $T_2 > T$, let k_2 be the integer such that $0 \leq T_2 - k_2 s < s$. Then, we have

$$|g(T_2,x)-g(T,x)| \leq \sum_{i=k}^{k_2+1} \int_{(i-1)s}^{is} |h(\tau,x)| d\tau \leq \sum_{i=1}^{k_2+1} \frac{1}{q(s,\epsilon)^i} C(s)s\epsilon_0 \leq \frac{1}{q(s,\epsilon)^k} \frac{1}{1-q(s,\epsilon)} C(s)s\epsilon_0$$

Therefore, there exists K_2 such that if T and T_2 greater than $(K_2+1)s$, we have

$$|g(T_2,x) - g(T,x)| \leq \epsilon_1$$

By the same way, we can get

$$|\nabla(g(T_2, x) - g(T, x))| \leq \epsilon_1$$

and

$$|\nabla^2(g(T_2, x) - g(T, x))| \leq \epsilon_1$$

This implies the convergence of the normalized Ricci flow in the weighted space. □ Actually, from the proof, we see that actually the proof just give us a way to generalized the time independent semigroup theory into time dependent semigroup theory.

4 Stability of Asymptotically hyperbolic Einstein manifolds

Theorem 4.1 *Let (M^{n+1}, g_+) be an asymptotically hyperbolic Einstein manifolds with nondegeneracy $\lambda > 0$ and regularity $C^{2,\alpha}$ and (M^{n+1}, g) be another asymptotically hyperbolic Einstein manifolds. Then, for any $\delta \in (0, n)$, there exists $\epsilon_0(\lambda) > 0$, such that if $|g - g_+| \leq \epsilon_0 e^{-\delta d(x_0, x)}$, Then the Ricci DeTurck flow with the initial g has the long time existence and*

$$\lim_{t \to \infty} |g - g_+|_{0,0,\delta} = 0$$

Proof : First, the long time existence is due to lemma 3.1. In fact $|g - g_+| \leq \epsilon_0 e^{-\delta d(x_0, x)}$ just implies that $|Ric(g) + ng| \leq \epsilon_0 e^{-\delta d(x_0, x)}$. Now, we will show

$$\lim_{t \to \infty} |g - g_+|_{0,0,\delta} = 0$$

Consider the Ricci-DeTurck flow

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}(g(t)) + \nabla_i W_j + \nabla_j W_i - 2(n-1) g_{ij}$$

where $W_j = g^{ll_1} g_{jk} \left(\Gamma^k_{ll_1}(g(t)) - \Gamma^k_{ll_1}(g_+) \right)$ and ∇ is the covariant derivative with respect to $g(t)$.

Let $h_{ij}(t, x) = g_{ij}(t, x) - g_{+ij}$. Then the Ricci-DeTurck flow is equivalent to the following flow

$$\frac{\partial}{\partial t} h_{ij} = \Delta_L h_{ij} - 2(n-1) h_{ij} + Q_{ij}(t, x)$$

where Δ_L is the **Lichnerowicz** Laplacian operator with respect to g_+ and the high order term Q is

$$Q_{ij}(t, x) = g * g^{-1} * g_+ * g_+^{-1} * \nabla h * \nabla h + g * g^{-1} * g_+ * g_+^{-1} * \nabla^2 h * h$$

For details of the computation, see the appendix. Then, the solution of the above Ricci DeTurck flow is

$$h(t,x) = S(t)(h(0,x)) + \int_0^t S(t-\tau)Q(\tau,x)d\tau$$

By the parabolic estimate, we have

$$|Q|_{0,0,\delta} \leq C(\epsilon_0)|h|_{0,0,\delta}$$

Therefore, we have

$$|h(t,x)|_{0,0,\delta} \leq e^{-\lambda_{\epsilon_0} t}|h(0,x)|_{0,0,\delta} + C(\epsilon_0)\int_0^t e^{-\lambda_{\epsilon_0}(t-\tau)}|h(\tau,x)|^2_{0,0,\delta}d\tau$$

let $G(t,x) = \max_{\tau\in[t-l,t]} e^{\lambda_\epsilon \tau}\left|h^K(\tau,x)\right|_{0,0,\delta}$. Then, we have

$$G(t,x) \leq |h(0,x)| + C(\epsilon_0)|G(t,x)|^2$$

Therefore,

$$|h(t,x)|_{0,0,\delta} \leq C(\epsilon_0)e^{-\lambda_{\epsilon_0} t}$$

Therefore, we the convergence. □

5 Perturbation existence recovered by Ricci flow

Suppose that $(\mathcal{M}^n, g)$ is a conformally compact Einstein manifold with the conformal infinity $(\partial\mathcal{M}, [\hat{g}])$. Suppose that r is the geodesic defining function associated with the conformal representative $\hat{g} \in [\hat{g}]$ on $\partial\mathcal{M}$. Then the metric expansion is given as follows (cf. [FG]):

$$\begin{aligned} g_r &= \hat{g} + g^{(2)}r^2 + \cdots + g^{(n-3)}r^{n-3} + hr^{n-1}\log r + g^{(n-1)}r^{n-1} + \cdots \\ &= \hat{g} + g^{(2)}r^2 + \cdots + g^{(k)}r^k + t^{(k)}[g] \end{aligned}$$

for $0 \leq k \leq n-3$, when $n-1$ is even

$$\begin{aligned} g_r &= \hat{g} + g^{(2)}r^2 + \cdots + g^{(n-2)}r^{n-2} + g^{(n-1)}r^{n-1} + \cdots \\ &= \hat{g} + g^{(2)}r^2 + \cdots + g^{(k)}r^k + t^{(k)}[g] \end{aligned}$$

for $0 \leq k \leq n-2$, when $n-1$ is odd, where

- $g^{(2i)}$ for $2i < n-1$ are local invariants of $\left(\partial\mathcal{M}^{n-1}, \hat{g}\right)$
- h and $\operatorname{tr} g^{(n-1)}(n-1$ even $)$ are also local invariant of $\left(\partial\mathcal{M}^{n-1}, \hat{g}\right)$
- h and $g^{(n-1)}(n-1$ odd $)$ are trace free
- $g^{(n-1)}(n-1$ odd $)$ and trace-free part of $g^{(n-1)}(n-1$ even $)$ are nonlocal.

For instance,

$$g^{(2)} = -\frac{1}{n-3}\left(\hat{R}ic - \frac{\hat{R}}{2(n-2)}\hat{g}\right)$$

To construct a candidate to be the right initial metric to apply Theorem 4.6, whose conformal infinity is a perturbation of that of a given conformally compact Einstein metric g, we set

$$g_r^{k,\nu} = \hat{g}_\nu + g_\nu^{(2)} r^2 + \cdots + g_\nu^{(k)} r^k + t^{(k)}[g]$$

where $\hat{g}_\nu$ is a perturbation of $\hat{g}$, and $g_\nu^{(2i)} = g^{(2i)}\left[\hat{g}_\nu\right], 2i \leq k$, are corresponding curvature terms of $\hat{g}_\nu$ as given in the metric expansion in [FG]. Next let ϕ be a cut-off function of the variable r such that $\phi = 0$ when $r \geq \nu_2$ and $\phi = 1$ when $r \leq \nu_1$, where $\nu_1 < \nu_2$ are chosen later. We therefore have the candidate

$$g_{k,\nu}^{\phi} = r^{-2}\left(dr^2 + (1-\phi)g_r + \phi g_r^{k,\nu}\right)$$

Immediately we see that

$$\left\|g_{k,\nu}^{\phi} - g\right\|_g \leq C\left\|\hat{g}_\nu - \hat{g}\right\|_{C^k}$$

$$\left\|\Gamma_{ij}^{l}\left[g_{k,\nu}^{\phi}\right] - \Gamma_{ij}^{l}[g]\right\|_g \leq C\left\|\hat{g}_\nu - \hat{g}\right\|_{C^{k+1}}$$

$$\left\|\mathrm{Rm}\left[g_{k,\nu}^{\phi}\right] - \mathrm{Rm}[g]\right\|_g \leq C\left\|\hat{g}_\nu - \hat{g}\right\|_{C^{k+2}}$$

$$\left\|\nabla Rm\left[g_{k,\nu}^{\phi}\right] - \nabla Rm[g]\right\| \leq C\left\|\hat{g}_\nu - \hat{g}\right\|_{C^{k+2}}$$

Theorem 5.1 *Let $(\mathcal{M}^n, g)$ be a conformally compact Einstein manifold of regularity C^2 with a smooth conformal infinity $(\partial\mathcal{M}, [\hat{g}])$. Assume that g is of the non-degeneracy λ_0. Then, if a smooth metric $[\hat{g}_\nu]$ is a sufficiently small $C^{k+2}-$perturbation of $[\hat{g}]$, then there is a C^2 -conformally compact Einstein metric on $\mathcal{M}$ whose conformal infinity is $[\hat{g}_\nu]$*

Proof: First of all, from the above discussion, it is clear that $\hat{g}_{k,\nu}^{\phi}$ satisfies the Theorem 4.6.

Theorem 5.2 *Let $(\mathcal{M}^n, g), n \geq 5$, be a conformally compact Einstein manifold of regularity C^2 with a smooth conformal infinity $(\partial\mathcal{M}, [\hat{g}])$. with non-degeneracy. Then, for any smooth metric $\hat{g}_\nu$ on $\partial\mathcal{M}$, which is sufficiently $C^{2,\alpha}$ close to some $\hat{g} \in [\hat{g}]$ for any $\alpha \in (0,1)$, there is a conformally compact Einstein metric g_ν on $\mathcal{M}$ which can be C^2 conformally compactfied with the conformal infinity $[\hat{g}_\nu]$.*

6 Appendix

6.1 The variation of the connection

Let ∇ and $\tilde{\nabla}$ be the connections for the metrics g and $\tilde{g}$ respectively. Take the normal coordinate with respect to $\tilde{g}$ at the point p_0. Let $\{e_i\}_{i=1}^n$ be the coordinate frame of this normal coordinate. Then we have

$$\nabla_i e_j - \tilde{\nabla}_i e_j = \frac{1}{2} g^{kl}(\tilde{\nabla}_j g_{li} - \tilde{\nabla}_l g_{ij} + \tilde{\nabla}_i g_{lj}) \stackrel{d}{=} C_{ij}^k e_k$$

Then

$$(\nabla_i - \tilde{\nabla}_i) a^k = C_{ij}^k a^j$$

$$(\nabla_i - \tilde{\nabla}_i) h_{jk} = -C_{ij}^{j_1} h_{j_1 k} - C_{ik}^{k_1} h_{jk_1}$$

where a^k is a covariant one order tensor and h_{ij} is a contravaiant two order tensor.

6.2 The variation of the curvature

With the same condition of proceeding section, we have

$$R(e_i, e_j) a^k = -\nabla_i \nabla_j a^k + \nabla_j \nabla_i a^k + \nabla_{[e_i, e_j]} a^k = R_{ijk_1}^k a^{k_1}$$

Since

$$\nabla_i a^j = \tilde{\nabla}_i a^j + C_{ij_1}^j a^{j_1},$$

we have

$$\begin{aligned}
\nabla_i \nabla_j a^k =& \tilde{\nabla}_i \nabla_j a^k + C_{ik_1}^k \nabla_j a^{k_1} - C_{ij}^{j_1} \nabla_{j_1} a^k \\
=& \tilde{\nabla}_i (\tilde{\nabla}_j a^k + C_{jk_1}^k a^{k_1}) + C_{ik_1}^k (\tilde{\nabla}_j a^{k_1} + C_{jk_2}^{k_1} a^{k_2}) - C_{ij}^{j_1} (\tilde{\nabla}_{j_1} a^k + C_{j_1 k_1}^k a^{k_1}) \\
=& \tilde{\nabla}_i \tilde{\nabla}_j a^k + (\tilde{\nabla}_i C_{jk_1}^k) a^{k_1} + C_{jk_1}^k \tilde{\nabla}_i a^{k_1} + C_{ik_1}^k \tilde{\nabla}_j a^{k_1} + C_{ik_1}^k C_{jk_2}^{k_1} a^{k_2} \\
& - C_{ij}^{j_1} \tilde{\nabla}_{j_1} a^k - C_{ij}^{j_1} C_{j_1 k_1}^k a^{k_1}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\nabla_j \nabla_i a^k =& \tilde{\nabla}_j \tilde{\nabla}_i a^k + (\tilde{\nabla}_j C_{ik_1}^k) a^{k_1} + C_{ik_1}^k \tilde{\nabla}_j a^{k_1} + C_{jk_1}^k (\tilde{\nabla}_i a^{k_1}) + C_{j_1}^k C_{ik_2}^{k_1} a^{k_2} \\
& - C_{ji}^{i_1} \tilde{\nabla}_{i_1} a^k - C_{ji}^{i_1} C_{i_1 k_1}^k a^{k_1}
\end{aligned}$$

Therefore,

$$\begin{aligned}
R(e_i, e_j) a^k =& - \nabla_i \nabla_j a^k + \nabla_j \nabla_i a^k \\
=& - \tilde{\nabla}_i \tilde{\nabla}_j a^k + \tilde{\nabla}_j \tilde{\nabla}_i a^k + (-\tilde{\nabla}_i C_{jk_1}^k + \tilde{\nabla}_j C_{ik_1}^k) a^{k_1} \\
& + (-C_{ik_1}^k C_{jk_2}^{k_1} + C_{jk_1}^k C_{ik_2}^{k_1}) a^{k_2} + (C_{ij}^{j_1} C_{j_1 k_1}^k - C_{ji}^{i_1} C_{i_1 k_1}^k) a^{k_1} \\
=& \tilde{R}(e_i, e_j) a^k + (-\tilde{\nabla}_i C_{jk_1}^k + \tilde{\nabla}_j C_{ik_1}^k) a^{k_1} + (-C_{ik_1}^k C_{jk_2}^{k_1} + C_{jk_1}^k C_{ik_2}^{k_1}) a^{k_2} \\
=& R_{ijk_1}^k a^{k_1}
\end{aligned}$$

Take $a = a^k e_k = \delta_l^k e_k = e_l$. Then, we have

$$R_{ijl}^k = \tilde{R}_{ijl}^k + (-\tilde{\nabla}_i C_{jl}^k + \tilde{\nabla}_j C_{il}^k) + (-C_{ik_1}^k C_{jl}^{k_1} + C_{jk_1}^k C_{il}^{k_1})$$

Since

$$\begin{aligned}\tilde{\nabla}_i C_{jl}^k =& \frac{1}{2} g^{kk_1} [\tilde{\nabla}_i \tilde{\nabla}_l g_{k_1 j} - \tilde{\nabla}_i \tilde{\nabla}_{k_1} g_{jl} + \tilde{\nabla}_i \tilde{\nabla}_j g_{k_1 l}] \\ &+ \frac{1}{2} \tilde{\nabla}_i g^{kk_1} [\tilde{\nabla}_l g_{k_1 j} - \tilde{\nabla}_{k_1} g_{jl} + \tilde{\nabla}_j g_{k_1 l}]\end{aligned}$$

and

$$\begin{aligned}\tilde{\nabla}_j C_{il}^k =& \frac{1}{2} g^{kk_1} [\tilde{\nabla}_j \tilde{\nabla}_l g_{k_1 i} - \tilde{\nabla}_j \tilde{\nabla}_{k_1} g_{il} + \tilde{\nabla}_j \tilde{\nabla}_i g_{k_1 l}] \\ &+ \frac{1}{2} \tilde{\nabla}_j g^{kk_1} [\tilde{\nabla}_l g_{k_1 i} - \tilde{\nabla}_{k_1} g_{il} + \tilde{\nabla}_i g_{k_1 l}],\end{aligned}$$

we have

$$\begin{aligned}R_{ijl}^k =& \tilde{R}_{ijl}^k - \frac{1}{2} g^{kk_1} [\tilde{\nabla}_i \tilde{\nabla}_l g_{k_1 j} - \tilde{\nabla}_i \tilde{\nabla}_{k_1} g_{jl} + \tilde{\nabla}_i \tilde{\nabla}_j g_{k_1 l} \\ &- \tilde{\nabla}_j \tilde{\nabla}_l g_{k_1 i} + \tilde{\nabla}_j \tilde{\nabla}_{k_1} g_{il} - \tilde{\nabla}_j \tilde{\nabla}_i g_{k_1 l}] - \frac{1}{2} \tilde{\nabla}_i g^{kk_1} [\tilde{\nabla}_l g_{k_1 j} - \tilde{\nabla}_{k_1} g_{jl} + \tilde{\nabla}_j g_{k_1 l}] \\ &+ \frac{1}{2} \tilde{\nabla}_j g^{kk_1} [\tilde{\nabla}_l g_{k_1 i} - \tilde{\nabla}_{k_1} g_{il} + \tilde{\nabla}_i g_{k_1 l}] + (-C_{ik_1}^k C_{jl}^{k_1} + C_{jk_1}^k C_{il}^{k_1}).\end{aligned}$$

Furthermore, we have

$$\begin{aligned}R_{ijl}^k =& \tilde{R}_{ijl}^k - \frac{1}{2} \tilde{g}^{kk_1} [\tilde{\nabla}_i \tilde{\nabla}_l g_{k_1 j} - \tilde{\nabla}_i \tilde{\nabla}_{k_1} g_{jl} + \tilde{\nabla}_i \tilde{\nabla}_j g_{k_1 l} \\ &- \tilde{\nabla}_j \tilde{\nabla}_l g_{k_1 i} + \tilde{\nabla}_j \tilde{\nabla}_{k_1} g_{il} - \tilde{\nabla}_j \tilde{\nabla}_i g_{k_1 l}] - \frac{1}{2} \tilde{\nabla}_i g^{kk_1} [\tilde{\nabla}_l g_{k_1 j} - \tilde{\nabla}_{k_1} g_{jl} + \tilde{\nabla}_j g_{k_1 l}] \\ &+ \frac{1}{2} \tilde{\nabla}_j g^{kk_1} [\tilde{\nabla}_l g_{k_1 i} - \tilde{\nabla}_{k_1} g_{il} + \tilde{\nabla}_i g_{k_1 l}] + (-C_{ik_1}^k C_{jl}^{k_1} + C_{jk_1}^k C_{il}^{k_1}) \\ &- \frac{1}{2} (g^{kk_1} - \tilde{g}^{kk_1}) [\tilde{\nabla}_i \tilde{\nabla}_l g_{k_1 j} - \tilde{\nabla}_i \tilde{\nabla}_{k_1} g_{jl} + \tilde{\nabla}_i \tilde{\nabla}_j g_{k_1 l} \\ &- \tilde{\nabla}_j \tilde{\nabla}_l g_{k_1 i} + \tilde{\nabla}_j \tilde{\nabla}_{k_1} g_{il} - \tilde{\nabla}_j \tilde{\nabla}_i g_{k_1 l}]\end{aligned}$$

Therefore, we have the Ricci curvature is

$$\begin{aligned}R_{il} =& \tilde{R}_{il} - \frac{1}{2} \tilde{g}^{jk_1} [\tilde{\nabla}_i \tilde{\nabla}_l g_{k_1 j} - \tilde{\nabla}_i \tilde{\nabla}_{k_1} g_{jl} + \tilde{\nabla}_i \tilde{\nabla}_j g_{k_1 l} \\ &- \tilde{\nabla}_j \tilde{\nabla}_l g_{k_1 i} + \tilde{\nabla}_j \tilde{\nabla}_{k_1} g_{il} - \tilde{\nabla}_j \tilde{\nabla}_i g_{k_1 l}] - \frac{1}{2} \tilde{\nabla}_i g^{jk_1} [\tilde{\nabla}_l g_{k_1 j} - \tilde{\nabla}_{k_1} g_{jl} + \tilde{\nabla}_j g_{k_1 l}] \\ &+ \frac{1}{2} \tilde{\nabla}_j g^{jk_1} [\tilde{\nabla}_l g_{k_1 i} - \tilde{\nabla}_{k_1} g_{il} + \tilde{\nabla}_i g_{k_1 l}] + (-C_{ik_1}^j C_{jl}^{k_1} + C_{jk_1}^j C_{il}^{k_1}) \\ &- \frac{1}{2} (g^{jk_1} - \tilde{g}^{jk_1}) [\tilde{\nabla}_i \tilde{\nabla}_l g_{k_1 j} - \tilde{\nabla}_i \tilde{\nabla}_{k_1} g_{jl} + \tilde{\nabla}_i \tilde{\nabla}_j g_{k_1 l} \\ &- \tilde{\nabla}_j \tilde{\nabla}_l g_{k_1 i} + \tilde{\nabla}_j \tilde{\nabla}_{k_1} g_{il} - \tilde{\nabla}_j \tilde{\nabla}_i g_{k_1 l}]\end{aligned}$$

Let

$$
\begin{aligned}
Q =& -\frac{1}{2}\tilde{\nabla}_i g^{jk_1}[\tilde{\nabla}_l g_{k_1 j} - \tilde{\nabla}_{k_1} g_{jl} + \tilde{\nabla}_j g_{k_1 l}] + \frac{1}{2}\tilde{\nabla}_j g^{jk_1}[\tilde{\nabla}_l g_{k_1 i} - \tilde{\nabla}_{k_1} g_{il} + \tilde{\nabla}_i g_{k_1 l}] \\
&+ (-C^j_{ik_1} C^{k_1}_{jl} + C^j_{jk_1} C^{k_1}_{il}) - \frac{1}{2}(g^{jk_1} - \tilde{g}^{jk_1})[\tilde{\nabla}_i \tilde{\nabla}_l g_{k_1 j} - \tilde{\nabla}_i \tilde{\nabla}_{k_1} g_{jl} + \tilde{\nabla}_i \tilde{\nabla}_j g_{k_1 l} \\
&- \tilde{\nabla}_j \tilde{\nabla}_l g_{k_1 i} + \tilde{\nabla}_j \tilde{\nabla}_{k_1} g_{il} - \tilde{\nabla}_j \tilde{\nabla}_i g_{k_1 l}] \\
=& g * g^{-1} * \tilde{g} * \tilde{g}^{-1} * (\tilde{\nabla} g) * (\tilde{\nabla} g) + g * g^{-1} * \tilde{g} * \tilde{g}^{-1} * (g - \tilde{g}) * \tilde{\nabla}^2 g
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
R_{il} =& \tilde{R}_{il} - \frac{1}{2}\tilde{g}^{jk_1}[\tilde{\nabla}_i \tilde{\nabla}_l g_{k_1 j} - \tilde{\nabla}_i \tilde{\nabla}_{k_1} g_{jl} + \tilde{\nabla}_i \tilde{\nabla}_j g_{k_1 l} \\
&- \tilde{\nabla}_j \tilde{\nabla}_l g_{k_1 i} + \tilde{\nabla}_j \tilde{\nabla}_{k_1} g_{il} - \tilde{\nabla}_j \tilde{\nabla}_i g_{k_1 l}] + Q
\end{aligned}
$$

And

$$
R_{il} = \tilde{R}_{il} - \frac{1}{2}\tilde{\Delta} g_{il} - \frac{1}{2}\tilde{\nabla}_i \tilde{\nabla}_l \tilde{g}^{jk_1} g_{k_1 j} - \frac{1}{2}\tilde{g}^{jk_1}(-\tilde{\nabla}_j \tilde{\nabla}_l g_{k_1 i} - \tilde{\nabla}_j \tilde{\nabla}_i g_{k_1 l}) + Q
$$

Since

$$
\begin{aligned}
\tilde{\nabla}_j \tilde{\nabla}_l g_{k_1 i} &= \tilde{\nabla}_l \tilde{\nabla}_j g_{k_1 i} + \tilde{R}^{k_2}_{jlk_1} g_{k_2 i} + \tilde{R}^{i_1}_{jli} g_{k_1 i_1} \\
\tilde{\nabla}_j \tilde{\nabla}_i g_{k_1 l} &= \tilde{\nabla}_i \tilde{\nabla}_j g_{k_1 l} + \tilde{R}^{k_2}_{jik_1} g_{k_2 l} + \tilde{R}^{l_1}_{jil} g_{k_1 l_1},
\end{aligned}
$$

$$
\begin{aligned}
R_{il} =& \tilde{R}_{il} - \frac{1}{2}\tilde{\Delta} g_{il} - \frac{1}{2}\tilde{\nabla}_i \tilde{\nabla}_l \tilde{g}^{jk_1} g_{k_1 j} + Q \\
&+ \frac{1}{2}\tilde{g}^{jk_1}[\tilde{\nabla}_l \tilde{\nabla}_j g_{k_1 i} + \tilde{\nabla}_i \tilde{\nabla}_j g_{k_1 l} + \tilde{R}^{k_2}_{jlk_1} g_{k_2 i} + \tilde{R}^{i_1}_{jli} g_{k_1 i_1} + \tilde{R}^{k_2}_{jik_1} g_{k_2 l} + \tilde{R}^{l_1}_{jil} g_{k_1 l_1}]
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
R_{il} =& \tilde{R}_{il} - \frac{1}{2}[\tilde{\Delta} g_{il} - \tilde{R}^{k_2}_l g_{k_2 i} - \tilde{R}^{k_2}_i g_{k_2 l} - 2\tilde{g}^{jk_1}\tilde{R}^{i_1}_{jli} g_{k_1 i_1}] \\
&- \frac{1}{2}[\tilde{\nabla}_i \tilde{\nabla}_l \tilde{g}^{jk_1} g_{k_1 j} - \tilde{g}^{jk_1}\tilde{\nabla}_l \tilde{\nabla}_j g_{k_1 i} - \tilde{g}^{jk_1}\tilde{\nabla}_i \tilde{\nabla}_j g_{k_1 l}] + Q
\end{aligned}
$$

$$
\begin{aligned}
R_{il} =& \tilde{R}_{il} - \frac{1}{2}[\tilde{\Delta} g_{il} - \tilde{g}^{jk_1}\tilde{R}_{lj} g_{k_1 i} - \tilde{g}^{jk_1}\tilde{R}_{ij} g_{k_1 l} - 2\tilde{g}^{jk_1}\tilde{R}^{i_1}_{jli} g_{k_1 i_1}] \\
&- \frac{1}{2}[\tilde{\nabla}_i \tilde{\nabla}_l \tilde{g}^{jk_1} g_{k_1 j} - \tilde{g}^{jk_1}\tilde{\nabla}_l \tilde{\nabla}_j g_{k_1 i} - \tilde{g}^{jk_1}\tilde{\nabla}_i \tilde{\nabla}_j g_{k_1 l}] + Q
\end{aligned}
$$

$$
\begin{aligned}
R_{il} =& \tilde{R}_{il} - \frac{1}{2}[\tilde{\Delta} g_{il} - \tilde{g}^{jk_1}\tilde{R}_{lj} g_{k_1 i} - \tilde{g}^{jk_1}\tilde{R}_{ij} g_{k_1 l} - 2\tilde{g}^{jk_1}\tilde{g}^{i_1 i_2}\tilde{R}_{jlii_2} g_{k_1 i_1}] \\
&- \frac{1}{2}[\tilde{\nabla}_i \tilde{\nabla}_l \tilde{g}^{jk_1} g_{k_1 j} - \tilde{g}^{jk_1}\tilde{\nabla}_l \tilde{\nabla}_j g_{k_1 i} - \tilde{g}^{jk_1}\tilde{\nabla}_i \tilde{\nabla}_j g_{k_1 l}] + Q
\end{aligned}
$$

$$
\begin{aligned}
R_{il} =& \tilde{R}_{il} - \frac{1}{2}[\tilde{\Delta} g_{il} - \tilde{g}^{jk_1}\tilde{R}_{lj}g_{k_1 i} - \tilde{g}^{jk_1}\tilde{R}_{ij}g_{k_1 l} + 2\tilde{g}^{jk_1}\tilde{g}^{i_1 i_2}\tilde{R}_{ii_2 lj}g_{k_1 i_1}] \\
&- \frac{1}{2}[\tilde{\nabla}_i\tilde{\nabla}_l\tilde{g}^{jk_1}g_{k_1 j} - \tilde{g}^{jk_1}\tilde{\nabla}_l\tilde{\nabla}_j g_{k_1 i} - \tilde{g}^{jk_1}\tilde{\nabla}_i\tilde{\nabla}_j g_{k_1 l}] + Q
\end{aligned}
$$

Now, define the **Lichnerowicz Laplacian** as

$$\tilde{\Delta}_L g_{il} = \tilde{\Delta} g_{il} - \tilde{g}^{jk_1}\tilde{R}_{lj}g_{k_1 i} - \tilde{g}^{jk_1}\tilde{R}_{ij}g_{k_1 l} + 2\tilde{g}^{jk_1}\tilde{g}^{i_1 i_2}\tilde{R}_{ii_2 lj}g_{k_1 i_1}$$

Therefore, the Ricci curvature can be written as

$$R_{il} = \tilde{R}_{il} - \frac{1}{2}[\tilde{\Delta}_L g_{il} + \tilde{\nabla}_i\tilde{\nabla}_l\tilde{g}^{jk_1}g_{k_1 j} - \tilde{g}^{jk_1}\tilde{\nabla}_l\tilde{\nabla}_j g_{k_1 i} - \tilde{g}^{jk_1}\tilde{\nabla}_i\tilde{\nabla}_j g_{k_1 l}] + Q$$

Moreover, let $h = g - \tilde{g}$. Then, we have

$$R_{il} = \tilde{R}_{il} - \frac{1}{2}[\tilde{\Delta}_L h_{il} + \tilde{\nabla}_i\tilde{\nabla}_l\tilde{g}^{jk_1}h_{k_1 j} - \tilde{g}^{jk_1}\tilde{\nabla}_l\tilde{\nabla}_j h_{k_1 i} - \tilde{g}^{jk_1}\tilde{\nabla}_i\tilde{\nabla}_j h_{k_1 l}] + Q$$

We call

$$-\frac{1}{2}[\tilde{\Delta}_L h_{il} + \tilde{\nabla}_i\tilde{\nabla}_l\tilde{g}^{jk_1}h_{k_1 j} - \tilde{g}^{jk_1}\tilde{\nabla}_l\tilde{\nabla}_j h_{k_1 i} - \tilde{g}^{jk_1}\tilde{\nabla}_i\tilde{\nabla}_j h_{k_1 l}]$$

the **Linearization of Ricci curvature** at $\tilde{g}$ and denote it as $L_{\tilde{g}}(h)$

Remark 6.1 *In some textbook, the Riemannian curvature is defined as*

$$R(e_i, e_j)e_k = \nabla_i\nabla_j e_k - \nabla_j\nabla_i e_k - \nabla_{[e_i,e_j]}e_k$$

And Ricci curvature is defined as

$$R_{jk} = \sum_{l=1}^{n} R^l_{ljk}$$

where n is the dimension of the manifold. (Our definition about Ricci curvature is $R_{ik} = \sum_{j=1}^{n} R^j_{ijk}$) Comparing with our definition, the difference of Riemannian curvature is just a negative sign. And the Ricci curvature of these two definitions are exactly same.

Non-elliptic term : There is an non-elliptic term which is

$$\begin{aligned}
&\frac{1}{2}[\tilde{\nabla}_i\tilde{\nabla}_l\tilde{g}^{jk_1}h_{k_1j} - \tilde{g}^{jk_1}\tilde{\nabla}_l\tilde{\nabla}_j h_{k_1i} - \tilde{g}^{jk_1}\tilde{\nabla}_i\tilde{\nabla}_j h_{k_1l}]\\
=&\frac{1}{2}\tilde{g}^{jk_1}[\tilde{\nabla}_i\tilde{\nabla}_l h_{k_1j} - \tilde{\nabla}_l\tilde{\nabla}_j h_{k_1i} - \tilde{\nabla}_i\tilde{\nabla}_j h_{k_1l}]\\
=&\frac{1}{2}\tilde{g}^{jk_1}[\frac{1}{2}(\tilde{\nabla}_i\tilde{\nabla}_l h_{k_1j} - 2\tilde{\nabla}_l\tilde{\nabla}_j h_{k_1i}) + \frac{1}{2}((\tilde{\nabla}_i\tilde{\nabla}_l h_{k_1j} - 2\tilde{\nabla}_i\tilde{\nabla}_j h_{k_1l})]\\
=&\frac{1}{2}\tilde{g}^{jk_1}[\frac{1}{2}\tilde{\nabla}_l(-\tilde{\nabla}_j h_{k_1i} + \tilde{\nabla}_i h_{k_1j} - \tilde{\nabla}_{k_1}h_{ji}) + \frac{1}{2}\tilde{\nabla}_i(-\tilde{\nabla}_j h_{k_1l} + \tilde{\nabla}_l h_{k_1j} - \tilde{\nabla}_{k_1}h_{jl})]\\
&+ \frac{1}{2}\tilde{g}^{jk_1}[\tilde{R}^{k_2}_{ilk_1}h_{k_2j} + \tilde{R}^{j_1}_{ilj}h_{k_1j_1}]\\
=&-\frac{1}{2}\tilde{g}^{jk_1}[\frac{1}{2}\tilde{\nabla}_l(\tilde{\nabla}_j h_{k_1i} - \tilde{\nabla}_i h_{k_1j} + \tilde{\nabla}_{k_1}h_{ji}) + \frac{1}{2}\tilde{\nabla}_i(\tilde{\nabla}_j h_{k_1l} - \tilde{\nabla}_l h_{k_1j} + \tilde{\nabla}_{k_1}h_{jl})]\\
=&-\frac{1}{2}\tilde{g}^{jk_1}[\tilde{g}_{ii_1}\frac{1}{2}\tilde{\nabla}_l\tilde{g}^{i_1i_2}(\tilde{\nabla}_j h_{k_1i_2} - \tilde{\nabla}_{i_2}h_{k_1j} + \tilde{\nabla}_{k_1}h_{ji_2})\\
&+ \tilde{g}_{ll_1}\frac{1}{2}\tilde{\nabla}_i\tilde{g}^{l_1l_2}(\tilde{\nabla}_j h_{k_1l_2} - \tilde{\nabla}_{l_2}h_{k_1j} + \tilde{\nabla}_{k_1}h_{jl_2})]\\
=&-\frac{1}{2}\tilde{g}^{jk_1}[\tilde{g}_{ii_1}\tilde{\nabla}_l(\Gamma^{i_1}_{k_1j} - \tilde{\Gamma}^{i_1}_{k_1j}) + \tilde{g}_{ll_1}\tilde{\nabla}_i(\Gamma^{l_1}_{k_1j} - \tilde{\Gamma}^{l_1}_{k_1j})]
\end{aligned}$$

Then, we have

$$\frac{1}{2}[\tilde{\nabla}_i\tilde{\nabla}_l\tilde{g}^{jk_1}h_{k_1j} - \tilde{g}^{jk_1}\tilde{\nabla}_l\tilde{\nabla}_j h_{k_1i} - \tilde{g}^{jk_1}\tilde{\nabla}_i\tilde{\nabla}_j h_{k_1l}] = -\frac{1}{2}\tilde{g}^{jk_1}[\tilde{g}_{ii_1}\tilde{\nabla}_l C^{i_1}_{k_1j} + \tilde{g}_{ll_1}\tilde{\nabla}_i C^{l_1}_{k_1j}]$$

Now, let $V_i = \tilde{g}^{jk_1}\tilde{g}_{ii_1}C^{i_1}_{k_1j}$ and $V_l = \tilde{g}^{jk_1}\tilde{g}_{ll_1}C^{l_1}_{k_1j}$. Then, we have

$$\frac{1}{2}[\tilde{\nabla}_i\tilde{\nabla}_l\tilde{g}^{jk_1}h_{k_1j} - \tilde{g}^{jk_1}\tilde{\nabla}_l\tilde{\nabla}_j h_{k_1i} - \tilde{g}^{jk_1}\tilde{\nabla}_i\tilde{\nabla}_j h_{k_1l}] = -\frac{1}{2}[\tilde{\nabla}_l V_i + \tilde{\nabla}_i V_l]$$

Gauge term : Consider

$$W_i = g^{jk_1}g_{ii_1}C^{i_1}_{k_1j} \quad and \quad W_l = g^{jk_1}g_{ll_1}C^{l_1}_{k_1j}.$$

Gauge term refers to

$$\frac{1}{2}[\nabla_l W_i + \nabla_i W_l]$$

Then,

$$\begin{aligned}
&\frac{1}{2}[\nabla_l W_i + \nabla_i W_l]\\
=&\frac{1}{2}[\tilde{\nabla}_l W_i + \tilde{\nabla}_i W_l - C^{i_1}_{li}W_{i_1} - C^{l_1}_{il}W_{l_1}]\\
=&\frac{1}{2}[\tilde{\nabla}_l V_i + \tilde{\nabla}_i V_l + \tilde{\nabla}_l(W_i - V_i) + \tilde{\nabla}_i(W_l - V_l) - C^{i_1}_{li}W_{i_1} - C^{l_1}_{il}W_{l_1}]\\
=&\frac{1}{2}[\tilde{\nabla}_l V_i + \tilde{\nabla}_i V_l] + \frac{1}{2}[\tilde{\nabla}_l(W_i - V_i) + \tilde{\nabla}_i(W_l - V_l)] - \frac{1}{2}[C^{i_1}_{li}W_{i_1} + C^{l_1}_{il}W_{l_1}]
\end{aligned}$$

where

$$
\begin{aligned}
&\frac{1}{2}[\tilde{\nabla}_l(W_i - V_i) + \tilde{\nabla}_i(W_l - V_l)] - \frac{1}{2}[C^{i_1}_{li}W_{i_1} + C^{l_1}_{il}W_{l_1}]\\
=&\frac{1}{2}[(\tilde{\nabla}_l g^{jk_1})g_{ii_1}C^{i_1}_{k_1j} + g^{jk_1}(\tilde{\nabla}_l g_{ii_1})C^{i_1}_{k_1j} + (\tilde{\nabla}_i g^{jk_1})g_{ll_1}C^{l_1}_{k_1j} + g^{jk_1}(\tilde{\nabla}_i g_{ll_1}C^{l_1}_{k_1j})]\\
&+ \frac{1}{2}[(g^{jk_1}g_{ii_1} - \tilde{g}^{jk_1}\tilde{g}_{ii_1})\tilde{\nabla}_l C^{i_1}_{k_1j} + (g^{jk_1}g_{ll_1} - \tilde{g}^{jk_1}\tilde{g}_{ll_1})\tilde{\nabla}_i C^{l_1}_{k_1j}] - \frac{1}{2}[C^{i_1}_{li}W_{i_1} + C^{l_1}_{il}W_{l_1}]\\
=&\frac{1}{2}[(\tilde{\nabla}_l g^{jk_1})g_{ii_1}C^{i_1}_{k_1j} + g^{jk_1}(\tilde{\nabla}_l g_{ii_1})C^{i_1}_{k_1j} + (\tilde{\nabla}_i g^{jk_1})g_{ll_1}C^{l_1}_{k_1j} + \frac{1}{2}g^{jk_1}(\tilde{\nabla}_i g_{ll_1}C^{l_1}_{k_1j})]\\
&+ \frac{1}{2}g^{jk_1}(g_{ii_1} - \tilde{g}_{ii_1})\tilde{\nabla}_l C^{i_1}_{k_1j} + (g^{jk_1} - \tilde{g}^{jk_1})\tilde{g}_{ii_1}\tilde{\nabla}_l C^{i_1}_{k_1j}\\
&+ \frac{1}{2}g^{jk_1}(g_{ll_1}\tilde{g}_{ll_1})\tilde{\nabla}_i C^{l_1}_{k_1j} + \frac{1}{2}(g^{jk_1} - \tilde{g}^{jk_1})\tilde{g}_{ll_1}\tilde{\nabla}_i C^{l_1}_{k_1j}\\
&- \frac{1}{2}[C^{i_1}_{li}W_{i_1} + C^{l_1}_{il}W_{l_1}]
\end{aligned}
$$

Therefore,

$$\frac{1}{2}\left[\nabla_l W_i + \nabla_i W_l\right] = \frac{1}{2}[\tilde{\nabla}_l V_i + \tilde{\nabla}_i V_l] + Q_1$$

where

$$
\begin{aligned}
Q_1 =&\frac{1}{2}[(\tilde{\nabla}_l g^{jk_1})g_{ii_1}C^{i_1}_{k_1j} + g^{jk_1}(\tilde{\nabla}_l g_{ii_1})C^{i_1}_{k_1j} + (\tilde{\nabla}_i g^{jk_1})g_{ll_1}C^{l_1}_{k_1j} + \frac{1}{2}g^{jk_1}(\tilde{\nabla}_i g_{ll_1}C^{l_1}_{k_1j})]\\
&+ \frac{1}{2}g^{jk_1}(g_{ii_1} - \tilde{g}_{ii_1})\tilde{\nabla}_l C^{i_1}_{k_1j} + (g^{jk_1} - \tilde{g}^{jk_1})\tilde{g}_{ii_1}\tilde{\nabla}_l C^{i_1}_{k_1j}\\
&+ \frac{1}{2}g^{jk_1}(g_{ll_1} - \tilde{g}_{ll_1})\tilde{\nabla}_i C^{l_1}_{k_1j} + \frac{1}{2}(g^{jk_1} - \tilde{g}^{jk_1})\tilde{g}_{ll_1}\tilde{\nabla}_i C^{l_1}_{k_1j}\\
&- \frac{1}{2}[C^{i_1}_{li}W_{i_1} + C^{l_1}_{il}W_{l_1}]\\
=&g * g^{-1} * \tilde{g} * \tilde{g}^{-1} * (\tilde{\nabla} g) * (\tilde{\nabla} g) + g * g^{-1} * \tilde{g} * \tilde{g}^{-1} * (g - \tilde{g}) * (\tilde{\nabla}^2 g)
\end{aligned}
$$

Therefore,

$$Q + Q_1 = g * g^{-1} * \tilde{g} * \tilde{g}^{-1} * (\tilde{\nabla} g) * (\tilde{\nabla} g) + g * g^{-1} * \tilde{g} * \tilde{g}^{-1} * (g - \tilde{g}) * (\tilde{\nabla}^2 g)$$

Ricci-DeTurk term : Consider

$$Ric(g) - \frac{1}{2}\left[\nabla_l W_i + \nabla_i W_l\right]$$

Then,

$$Ric(g) - \frac{1}{2}\left[\nabla_l W_i + \nabla_i W_l\right] = Ric(\tilde{g}) - \frac{1}{2}\tilde{\Delta}_L(g - \tilde{g}) + Q + Q_1$$

The linearization of the gauge term at different metrics : Let $\bar{g}$ be another metrics with Christoffel symbol Γ. Then

$$
\begin{aligned}
W_i =& g^{jk_1} g_{ii_1} C^{i_1}_{k_1 j} = g^{jk_1} g_{ii_1}[\Gamma^{i_1}_{k_1 j} - \tilde{\Gamma}^{i_1}_{k_1 j}] = g^{jk_1} g_{ii_1}[(\Gamma^{i_1}_{k_1 j} - \bar{\Gamma}^{i_1}_{k_1 j}) + (\bar{\Gamma}^{i_1}_{k_1 j} - \tilde{\Gamma}^{i_1}_{k_1 j})] \\
=& g^{jk_1}[\bar{\nabla}_j g_{k_1 i} - \frac{1}{2}\bar{\nabla}_i g_{k_1 j} - \bar{\nabla}_j \tilde{g}_{k_1 i} + \frac{1}{2}\bar{\nabla}_i \tilde{g}_{k_1 j}] = g^{-1} * (\bar{\nabla} g + \bar{\nabla}\tilde{g})
\end{aligned}
$$

Let

$$
\bar{V}_i = \bar{g}^{jk_1} \bar{g}_{ii_1}(\bar{\Gamma}^{i_1}_{k_1 j} - \tilde{\Gamma}^{i_1}_{k_1 j}).
$$

and

$$
V_i = \bar{g}^{jk_1} \bar{g}_{ii_1}(\Gamma^{i_1}_{k_1 j} - \bar{\Gamma}^{i_1}_{k_1 j})
$$

Similarly, we have

$$
\begin{aligned}
W_l =& g^{jk_1} g_{ll_1}[(\Gamma^{l_1}_{k_1 j} - \bar{\Gamma}^{l_1}_{k_1 j}) + (\bar{\Gamma}^{l_1}_{k_1 j} - \tilde{\Gamma}^{l_1}_{k_1 j})] \\
=& g^{jk_1}[\bar{\nabla}_j g_{k_1 l} - \frac{1}{2}\bar{\nabla}_l g_{k_1 j} - \bar{\nabla}_j \tilde{g}_{k_1 l} + \frac{1}{2}\bar{\nabla}_l \tilde{g}_{k_1 j}] = g^{-1} * (\bar{\nabla} g + \bar{\nabla}\tilde{g})
\end{aligned}
$$

And

$$
\bar{V}_l = \bar{g}^{jk_1} \bar{g}_{ll_1}(\bar{\Gamma}^{l_1}_{k_1 j} - \tilde{\Gamma}^{l_1}_{k_1 j})
$$

and

$$
V_l = \bar{g}^{jk_1} \bar{g}_{ll_1}(\Gamma^{l_1}_{k_1 j} - \bar{\Gamma}^{l_1}_{k_1 j})
$$

More specifically, the gauge term can be written as

$$
\begin{aligned}
& \frac{1}{2}[\nabla_l W_i + \nabla_i W_l] \\
=& \frac{1}{2}[\bar{\nabla}_l W_i + \bar{\nabla}_i W_l + (\nabla_l - \bar{\nabla}_l) W_i + (\nabla_i - \bar{\nabla}_i) W_l] \\
=& \frac{1}{2}[\bar{\nabla}_l V_i + \bar{\nabla}_i V_l + (\nabla_l - \bar{\nabla}_l) W_i + (\nabla_i - \bar{\nabla}_i) W_l + \bar{\nabla}_l (W_i - V_i) + \bar{\nabla}_i (W_l - V_l)] \\
& \frac{1}{2}[\bar{\nabla}_l V_i + \bar{\nabla}_i V_l] + \frac{1}{2}[(\nabla_l - \bar{\nabla}_l) W_i + (\nabla_i - \bar{\nabla}_i) W_l] + \frac{1}{2}[\bar{\nabla}_l (W_i - \bar{V}_i - V_i) + \bar{\nabla}_i (W_l - \bar{V}_l - V_l)] \\
& + \frac{1}{2}[\bar{\nabla}_l \bar{V}_i + \bar{\nabla}_i \bar{V}_l]
\end{aligned}
$$

where

$$
\begin{aligned}
& \frac{1}{2}[(\nabla_l - \bar{\nabla}_l) W_i + (\nabla_i - \bar{\nabla}_i) W_l] \\
=& -\frac{1}{2}(\Gamma^{i_1}_{li} - \bar{\Gamma}^{i_1}_{li}) W_{i_1} - \frac{1}{2}(\Gamma^{l_1}_{il} - \bar{\Gamma}^{l_1}_{il}) W_{l_1} \\
=& -\frac{1}{4} g^{i_1 i_2}(\bar{\nabla}_l g_{ii_2} - \bar{\nabla}_{i_2} g_{li} + \bar{\nabla}_i g_{i_2 l}) W_{i_1} - \frac{1}{4} g^{l_1 l_2}(\bar{\nabla}_i g_{ll_2} - \bar{\nabla}_{l_2} g_{il} + \bar{\nabla}_l g_{il_2}) W_{l_1} \\
=& -\frac{1}{4} g^{i_1 i_2} g^{jk_1}(\bar{\nabla}_l g_{ii_2} - \bar{\nabla}_{i_2} g_{li} + \bar{\nabla}_i g_{i_2 l})(\bar{\nabla}_j g_{k_1 i_1} - \frac{1}{2}\bar{\nabla}_i g_{k_1 j} - \bar{\nabla}_j \tilde{g}_{k_1 i_1} + \frac{1}{2}\bar{\nabla}_{i_1} \tilde{g}_{k_1 j}) \\
& - \frac{1}{4} g^{l_1 l_2} g^{jk_1}(\bar{\nabla}_i g_{ll_2} - \bar{\nabla}_{l_2} g_{il} + \bar{\nabla}_l g_{il_2})(\bar{\nabla}_j g_{k_1 l_1} - \frac{1}{2}\bar{\nabla}_{l_1} g_{k_1 j} - \bar{\nabla}_j \tilde{g}_{k_1 l_1} + \frac{1}{2}\bar{\nabla}_{l_1} \tilde{g}_{k_1 j}) \\
=& g * g^{-1} * (\bar{\nabla} g) * (\bar{\nabla} g + \bar{\nabla}\tilde{g})
\end{aligned}
$$

and

$$
\begin{aligned}
&\bar{\nabla}_l(W_i-\bar{V}_i-V_i)\\
=&\bar{\nabla}_l\left[g^{jk_1}g_{ii_1}[(\Gamma^{i_1}_{k_1j}-\bar{\Gamma}^{i_1}_{k_1j})+(\bar{\Gamma}^{i_1}_{k_1j}-\tilde{\Gamma}^{i_1}_{k_1j})]-\bar{g}^{jk_1}\bar{g}_{ii_1}(\bar{\Gamma}^{i_1}_{k_1j}-\tilde{\Gamma}^{i_1}_{k_1j})-\bar{g}^{jk_1}\bar{g}_{ii_1}(\Gamma^{i_1}_{k_1j}-\bar{\Gamma}^{i_1}_{k_1j})\right]\\
=&\bar{\nabla}_l\left[g^{jk_1}g_{ii_1}(\Gamma^{i_1}_{k_1j}-\bar{\Gamma}^{i_1}_{k_1j})+g^{jk_1}g_{ii_1}(\bar{\Gamma}^{i_1}_{k_1j}-\tilde{\Gamma}^{i_1}_{k_1j})-\bar{g}^{jk_1}\bar{g}_{ii_1}(\bar{\Gamma}^{i_1}_{k_1j}-\tilde{\Gamma}^{i_1}_{k_1j})-\bar{g}^{jk_1}\bar{g}_{ii_1}(\Gamma^{i_1}_{k_1j}-\bar{\Gamma}^{i_1}_{k_1j})\right]\\
=&\bar{\nabla}_l\left[(g^{jk_1}-\bar{g}^{jk_1})g_{ii_1}(\Gamma^{i_1}_{k_1j}-\bar{\Gamma}^{i_1}_{k_{1,j}})+\bar{g}^{jk_1}(g_{ii_1}-\bar{g}_{ii_1})(\Gamma^{i_1}_{k_1j}-\bar{\Gamma}^{i_1}_{k_{1,j}})\right]\\
&+\bar{\nabla}_l\left[(g^{jk_1}-\bar{g}^{jk_1})g_{ii_1}(\bar{\Gamma}^{i_1}_{k_1j}-\tilde{\Gamma}^{i_1}_{k_1j})+\bar{g}^{jk_1}(g_{ii_1}-\bar{g}_{ii_1})(\bar{\Gamma}^{i_1}_{k_1j}-\tilde{\Gamma}^{i_1}_{k_1j})\right]\\
=&(\bar{\nabla}_lg^{jk_1})g_{ii_1}(\Gamma^{i_1}_{k_1j}-\bar{\Gamma}^{i_1}_{k_1j})+g^{jk_1}(\bar{\nabla}_lg_{ii_1})(\Gamma^{i_1}_{k_1j}-\bar{\Gamma}^{i_1}_{k_1j})\\
&+[(g^{jk_1}-\bar{g}^{jk_1})g_{ii_1}+\bar{g}^{jk_1}(g_{ii_1}-\bar{g}_{ii_1})]\bar{\nabla}_l(\Gamma^{i_1}_{k_1j}-\bar{\Gamma}^{i_1}_{k_{1,j}})\\
&+(\bar{\nabla}_lg^{jk_1})g_{ii_1}(\bar{\Gamma}^{i_1}_{k_1j}-\tilde{\Gamma}^{i_1}_{k_1j})+g^{jk_1}(\bar{\nabla}_lg_{ii_1})(\bar{\Gamma}^{i_1}_{k_1j}-\tilde{\Gamma}^{i_1}_{k_1j})\\
&+[(g^{jk_1}-\bar{g}^{jk_1})g_{ii_1}+\bar{g}^{jk_1}(g_{ii_1}-\bar{g}_{ii_1})]\bar{\nabla}_l(\bar{\Gamma}^{i_1}_{k_1j}-\tilde{\Gamma}^{i_1}_{k_1j})\\
=&g*g^{-1}*\tilde{g}*\tilde{g}^{-1}*\bar{g}*\bar{g}^{-1}*[(\bar{\nabla}g)*(\bar{\nabla}g)+(g-\bar{g})*(\bar{\nabla}^2g)\\
&+(\bar{\nabla}g)*(\bar{\nabla}\tilde{g})+(g-\bar{g})*\bar{\nabla}^2\tilde{g}+(g-\bar{g})*(\bar{\nabla}\tilde{g})*(\bar{\nabla}\tilde{g})]\\
=&g*g^{-1}*\tilde{g}*\tilde{g}^{-1}*\bar{g}*\bar{g}^{-1}*[\bar{\nabla}g*(\bar{\nabla}g+\bar{\nabla}\tilde{g})+(g-\bar{g})*(\bar{\nabla}^2g+\bar{\nabla}^2\tilde{g}+\bar{\nabla}\tilde{g}*\bar{\nabla}\tilde{g})]
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\bar{\nabla}_l\left(W_i-\bar{V}_i-V_i\right)+\bar{\nabla}_i\left(W_l-\bar{V}_l-V_l\right)\\
=&(\bar{\nabla}_lg^{jk_1})g_{ii_1}(\Gamma^{i_1}_{k_1j}-\bar{\Gamma}^{i_1}_{k_1j})+g^{jk_1}(\bar{\nabla}_lg_{ii_1})(\Gamma^{i_1}_{k_1j}-\bar{\Gamma}^{i_1}_{k_1j})\\
&+[(g^{jk_1}-\bar{g}^{jk_1})g_{ii_1}+\bar{g}^{jk_1}(g_{ii_1}-\bar{g}_{ii_1})]\bar{\nabla}_l(\Gamma^{i_1}_{k_1j}-\bar{\Gamma}^{i_1}_{k_{1,j}})\\
&+(\bar{\nabla}_lg^{jk_1})g_{ii_1}(\bar{\Gamma}^{i_1}_{k_1j}-\tilde{\Gamma}^{i_1}_{k_1j})+g^{jk_1}(\bar{\nabla}_lg_{ii_1})(\bar{\Gamma}^{i_1}_{k_1j}-\tilde{\Gamma}^{i_1}_{k_1j})\\
&+[(g^{jk_1}-\bar{g}^{jk_1})g_{ll_1}+\bar{g}^{jk_1}(g_{ll_1}-\bar{g}_{ll_1})]\bar{\nabla}_i(\bar{\Gamma}^{l_1}_{k_1j}-\tilde{\Gamma}^{l_1}_{k_{1,j}})\\
&+(\bar{\nabla}_ig^{jk_1})g_{ll_1}(\Gamma^{l_1}_{k_1j}-\bar{\Gamma}^{l_1}_{k_1j})+g^{jk_1}(\bar{\nabla}_ig_{ll_1})(\Gamma^{l_1}_{k_1j}-\bar{\Gamma}^{l_1}_{k_1j})\\
&+[(g^{jk_1}-\bar{g}^{jk_1})g_{ll_1}+\bar{g}^{jk_1}(g_{ll_1}-\bar{g}_{ll_1})]\bar{\nabla}_i(\Gamma^{l_1}_{k_1j}-\bar{\Gamma}^{l_1}_{k_{1,j}})\\
&+(\bar{\nabla}_ig^{jk_1})g_{ll_1}(\bar{\Gamma}^{l_1}_{k_1j}-\tilde{\Gamma}^{l_1}_{k_1j})+g^{jk_1}(\bar{\nabla}_ig_{ll_1})(\bar{\Gamma}^{l_1}_{k_1j}-\tilde{\Gamma}^{l_1}_{k_1j})\\
&+[(g^{jk_1}-\bar{g}^{jk_1})g_{ll_1}+\bar{g}^{jk_1}(g_{ll_1}-\bar{g}_{ll_1})]\bar{\nabla}_i(\bar{\Gamma}^{l_1}_{k_1j}-\tilde{\Gamma}^{l_1}_{k_1j})\\
=&g*g^{-1}*\tilde{g}*\tilde{g}^{-1}*\bar{g}*\bar{g}^{-1}*[\bar{\nabla}g*(\bar{\nabla}g+\bar{\nabla}\tilde{g})+(g-\bar{g})*(\bar{\nabla}^2g+\bar{\nabla}^2\tilde{g}+\bar{\nabla}\tilde{g}*\bar{\nabla}\tilde{g})]
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{1}{2}[\bar{\nabla}_l\bar{V}_i+\bar{\nabla}_i\bar{V}_l]=&\frac{1}{2}\bar{\nabla}_l[\bar{g}^{jk_1}\bar{g}_{ii_1}(\bar{\Gamma}^{i_1}_{k_1j}-\tilde{\Gamma}^{i_1}_{k_1j})]+\frac{1}{2}\bar{\nabla}_i[\bar{g}^{jk_1}\bar{g}_{ll_1}(\bar{\Gamma}^{l_1}_{k_1j}-\tilde{\Gamma}^{l_1}_{k_1j})]\\
=&\bar{g}*\bar{g}^{-1}*\tilde{g}*\tilde{g}^{-1}*\bar{\nabla}^2\tilde{g}
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\frac{1}{2}\left[\nabla_lW_i+\nabla_iW_l\right]=\frac{1}{2}\left[\bar{\nabla}_lV_i+\bar{\nabla}_iV_l\right]\\
&+g*g^{-1}*\tilde{g}*\tilde{g}^{-1}*\bar{g}*\bar{g}^{-1}*[\bar{\nabla}g*(\bar{\nabla}g+\bar{\nabla}\tilde{g})+(g-\bar{g})*(\bar{\nabla}^2g+\bar{\nabla}^2\tilde{g}+\bar{\nabla}\tilde{g}*\bar{\nabla}\tilde{g})+\bar{\nabla}^2\tilde{g}]
\end{aligned}
$$

The Linearization of Ricci-DeTurck term at different metrics : Let $\bar{g}$ be another metric. Then, the Ricci-DeTurck term can be written as

$$Ric(g) - \frac{1}{2}[\nabla_l W_i + \nabla_i W_l] = Ric(\bar{g}) - \frac{1}{2}\bar{\Delta}_L(g-\bar{g}) + Q_2$$

where

$$Q_2 = g*g^{-1}*\tilde{g}*\tilde{g}^{-1}*\bar{g}*\bar{g}^{-1}*[\bar{\nabla}g*(\bar{\nabla}g+\bar{\nabla}\tilde{g})+(g-\bar{g})*(\bar{\nabla}^2 g+\bar{\nabla}^2\tilde{g}+\bar{\nabla}\tilde{g}*\bar{\nabla}\tilde{g})+\bar{\nabla}^2\tilde{g}]$$

Now, let $h = g - \bar{g}$ and $\tilde{h} = \bar{g} - \tilde{g}$. We have

$$Ric(g) - \frac{1}{2}[\nabla_l W_i + \nabla_i W_l] = Ric(\bar{g}) - \frac{1}{2}\bar{\Delta}_L h + Q_2$$

where

$$Q_2 = g*g^{-1}*\tilde{g}*\tilde{g}^{-1}*\bar{g}*\bar{g}^{-1}*[\bar{\nabla}h*(\bar{\nabla}h+\bar{\nabla}\tilde{h})+h*(\bar{\nabla}^2 h+\bar{\nabla}^2\tilde{h}+\bar{\nabla}\tilde{h}*\bar{\nabla}\tilde{h})+\bar{\nabla}^2\tilde{h}]$$

6.3 The variation of Lichnerowicz Laplacian operator

Let $\tilde{\Delta}_L$ and $\bar{\Delta}_L$ be the Lichnerowicz Laplacian operator for the metrics $\tilde{g}$ and $\bar{g}$ respectively. And let

$$\tilde{\nabla}_j a^k - \bar{\nabla}_j a^k = C^k_{jk_2} a^{k_2}$$

and

$$C^k_{jk_2} = \tilde{g}^{kk_3}(\bar{\nabla}_j \tilde{g}_{k_2k_3} - \bar{\nabla}_{k_3}\tilde{g}_{jk_2} + \bar{\nabla}_{k_2}\tilde{g}_{jk_3})$$

Then we have

$$\begin{aligned}(\tilde{\Delta}_L - \bar{\Delta}_L)g_{il} =&(\tilde{\Delta} - \bar{\Delta})g_{il} - [\tilde{g}^{jk_1}\tilde{R}_{lj}g_{k_1i} - \bar{g}^{jk_1}\bar{R}_{lj}g_{k_1i}] - [\tilde{g}^{jk_1}\tilde{R}_{ij}g_{k_1l} - \bar{g}^{jk_1}\bar{R}_{ij}g_{k_1l}]\\ &+ 2[\tilde{g}^{jk_1}\tilde{g}^{i_1i_2}\tilde{R}_{ii_2lj}g_{k_1i_1} - \bar{g}^{jk_1}\bar{g}^{i_1i_2}\bar{R}_{ii_2lj}g_{k_1i_1}]\end{aligned}$$

where

$$\begin{aligned}(\tilde{\Delta} - \bar{\Delta})g_{il} =&\tilde{g}^{jk_1}\tilde{\nabla}_j\tilde{\nabla}_{k_1}g_{il} - \bar{g}^{jk_1}\bar{\nabla}_j\bar{\nabla}_{k_1}g_{il}\\ =&(\tilde{g}^{jk_1} - \bar{g}^{jk_1})\bar{\nabla}_j\bar{\nabla}_{k_1}g_{il} + \tilde{g}^{jk_1}(\tilde{\nabla}_j\tilde{\nabla}_{k_1}g_{il} - \bar{\nabla}_j\bar{\nabla}_{k_1}g_{il})\end{aligned}$$

and

$$\begin{aligned}\tilde{g}^{jk_1}(\tilde{\nabla}_j\tilde{\nabla}_{k_1}g_{il} - \bar{\nabla}_j\bar{\nabla}_{k_1}g_{il}) =&\tilde{g}^{jk_1}[\bar{\nabla}_j(\tilde{\nabla}_{k_1} - \bar{\nabla}_{k_1})g_{il} + (\tilde{\nabla}_j - \bar{\nabla}_j)\tilde{\nabla}_{k_1}g_{il}\\ &+ (\tilde{\nabla}_j - \bar{\nabla}_j)(\tilde{\nabla}_{k_1} - \bar{\nabla}_{k_1})g_{il} + (\tilde{\nabla}_j - \bar{\nabla}_j)\bar{\nabla}_{k_1}g_{il}]\\ =&\tilde{g} * \tilde{g}^{-1} * \bar{g} * \bar{g}^{-1} * [(\bar{\nabla}\tilde{g}) * \bar{\nabla}\tilde{g} + \bar{\nabla}^2\tilde{g}] * g\\ &+ \tilde{g} * \tilde{g}^{-1} * \bar{g} * \bar{g}^{-1} * (\bar{\nabla}\tilde{g}) * (\bar{\nabla}g)\\ &+ \tilde{g} * \tilde{g}^{-1} * \bar{g} * \bar{g}^{-1} * [(\bar{\nabla}\tilde{g}) * (\bar{\nabla}\tilde{g})] * g\\ &+ \tilde{g} * \tilde{g}^{-1} * \bar{g} * \bar{g}^{-1} * (\bar{\nabla}\tilde{g}) * \bar{\nabla}g\end{aligned}$$

Therefore,

$$
\begin{aligned}
(\tilde{\Delta}-\bar{\Delta})g_{il} =&(\tilde{g}^{jk_1}-\bar{g}^{jk_1})\bar{\nabla}_j\bar{\nabla}_{k_1}g_{il}\\
&+\tilde{g}*\tilde{g}^{-1}*\bar{g}*\bar{g}^{-1}*[(\bar{\nabla}\tilde{g})*\bar{\nabla}\tilde{g}+\bar{\nabla}^2\tilde{g}]*g\\
&+\tilde{g}*\tilde{g}^{-1}*\bar{g}*\bar{g}^{-1}*(\bar{\nabla}\tilde{g})*(\bar{\nabla}g)
\end{aligned}
$$

For the zero order term

$$
\begin{aligned}
&\tilde{g}^{jk_1}\tilde{R}_{ij}g_{k_1l}-\bar{g}^{jk_1}\bar{R}_{ij}g_{k_1l}=(\tilde{g}^{jk_1}-\bar{g}^{jk_1})\bar{R}_{ij}g_{k_1l}+\tilde{g}^{jk_1}(\tilde{R}_{ij}-\bar{R}_{ij})g_{k_1l}\\
=&(\tilde{g}^{jk_1}-\bar{g}^{jk_1})\bar{R}_{ij}g_{k_1l}+\tilde{g}*\tilde{g}^{-1}*\bar{g}*\bar{g}^{-1}*[\bar{\nabla}\tilde{g}*\bar{\nabla}\tilde{g}+(\tilde{g}-\bar{g})*\bar{\nabla}^2\tilde{g}]*g\\
=&\tilde{g}*\tilde{g}^{-1}*\bar{g}*\bar{g}^{-1}*[\bar{\nabla}\tilde{g}*\bar{\nabla}\tilde{g}+(\tilde{g}-\bar{g})*(\bar{\nabla}^2\tilde{g}+\bar{R})]*g
\end{aligned}
$$

By the same way, we can get that

$$
\begin{aligned}
&\tilde{g}^{jk_1}\tilde{g}^{i_1i_2}\tilde{R}_{ii_2lj}g_{k_1i_1}-\bar{g}^{jk_1}\bar{g}^{i_1i_2}\bar{R}_{ii_2lj}g_{k_1i_1}\\
=&(\tilde{g}^{jk_1}\tilde{g}^{i_1i_2}-\bar{g}^{jk_1}\bar{g}^{i_1i_2})\bar{R}_{ii_2lj}g_{k_1i_1}+\tilde{g}^{jk_1}\tilde{g}^{i_1i_2}(\tilde{R}_{ii_2lj}-\bar{R}_{ii_2lj})g_{k_1i_1}\\
=&[(\tilde{g}^{jk_1}-\bar{g}^{jk_1})\bar{g}^{i_1i_2}+\tilde{g}^{jk_1}(\tilde{g}^{i_1i_2}-\bar{g}^{i_1i_2})]\bar{R}_{ii_2lj}g_{k_1i_1}+\tilde{g}^{jk_1}\tilde{g}^{i_1i_2}(\tilde{R}_{ii_2lj}-\bar{R}_{ii_2lj})g_{k_1i_1}\\
=&\tilde{g}*\tilde{g}^{-1}*\bar{g}*\bar{g}^{-1}*[\bar{\nabla}\tilde{g}*\bar{\nabla}\tilde{g}+(\tilde{g}-\bar{g})*(\bar{\nabla}^2\tilde{g}+\bar{R})]*g
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
(\tilde{\Delta}_L-\bar{\Delta}_L)g =&\tilde{g}*\tilde{g}^{-1}*\bar{g}*\bar{g}^{-1}*[\bar{\nabla}\tilde{g}*\bar{\nabla}\tilde{g}+(\tilde{g}-\bar{g})*(\bar{\nabla}^2\tilde{g}+\bar{R})]*g\\
&+\tilde{g}*\tilde{g}^{-1}*\bar{g}*\bar{g}^{-1}*(\bar{\nabla}\tilde{g})*(\bar{\nabla}g)
\end{aligned}
$$

6.4 Curvature evolution equation

Consider the normalized Ricci flow

$$
\begin{cases}
\dfrac{d}{dt}g(t)=-2\left(\mathrm{Ric}_{g(t)}+(n-1)g(t)\right)\\
\quad g(0)=g_0
\end{cases}
$$

Let $h(t)=\mathrm{Ric}_{g(t)}+(n-1)g(t)$. We will induce the corresponding evolution equation of $h(t)$.

$$
\begin{aligned}
\frac{d}{dt}h(t) =&\frac{d}{dt}\mathrm{Ric}_{g(t)}+(n-1)\frac{d}{dt}g(t)\\
=&\frac{d}{dt}\mathrm{Ric}_{g(t)}-2(n-1)h(t)
\end{aligned}
$$

For simplicity, we will write $\mathrm{Ric}_{g(t)}$ as R_{il} and write the Riemannian curvature as R_{ijkl}. By the previous linearization of the Ricci curvature, we have

$$\begin{aligned} R_{il} =&\tilde{R}_{il} - \frac{1}{2}\tilde{g}^{jk_1}[\tilde{\nabla}_i\tilde{\nabla}_l g_{k_1j} - \tilde{\nabla}_i\tilde{\nabla}_{k_1}g_{jl} + \tilde{\nabla}_i\tilde{\nabla}_j g_{k_1l} \\ &- \tilde{\nabla}_j\tilde{\nabla}_l g_{k_1i} + \tilde{\nabla}_j\tilde{\nabla}_{k_1}g_{il} - \tilde{\nabla}_j\tilde{\nabla}_i g_{k_1l}] - \frac{1}{2}\tilde{\nabla}_i g^{jk_1}[\tilde{\nabla}_l g_{k_1j} - \tilde{\nabla}_{k_1}g_{jl} + \tilde{\nabla}_j g_{k_1l}] \\ &+ \frac{1}{2}\tilde{\nabla}_j g^{jk_1}[\tilde{\nabla}_l g_{k_1i} - \tilde{\nabla}_{k_1}g_{il} + \tilde{\nabla}_i g_{k_1l}] + (-C^j_{ik_1}C^{k_1}_{jl} + C^j_{jk_1}C^{k_1}_{il}) \\ &- \frac{1}{2}(g^{jk_1} - \tilde{g}^{jk_1})[\tilde{\nabla}_i\tilde{\nabla}_l g_{k_1j} - \tilde{\nabla}_i\tilde{\nabla}_{k_1}g_{jl} + \tilde{\nabla}_i\tilde{\nabla}_j g_{k_1l} \\ &- \tilde{\nabla}_j\tilde{\nabla}_l g_{k_1i} + \tilde{\nabla}_j\tilde{\nabla}_{k_1}g_{il} - \tilde{\nabla}_j\tilde{\nabla}_i g_{k_1l}] \end{aligned}$$

Therefore,

$$\frac{d}{dt}\mathrm{Ric}_{il} = -2\cdot\frac{1}{2}g^{jk_1}[\nabla_i\nabla_l h_{k_1j} - \nabla_i\nabla_{k_1}h_{jl} + \nabla_i\nabla_j h_{k_1l} - \nabla_j\nabla_l h_{k_1i} + \nabla_j\nabla_{k_1}h_{il} - \nabla_j\nabla_i h_{k_1l}]$$

Since

$$\begin{aligned} \nabla_j\nabla_l h_{k_1i} &= \nabla_l\nabla_j h_{k_1i} + R^{k_2}_{jlk_1}h_{k_2i} + R^{i_1}_{jli}h_{k_1i_1} \\ \nabla_j\nabla_i h_{k_1l} &= \nabla_i\nabla_j h_{k_1l} + R^{k_2}_{jik_1}h_{k_2l} + R^{l_1}_{jil}h_{k_1l_1}, \end{aligned}$$

we have

$$\begin{aligned} \frac{d}{dt}Ric_{il} =& -(-2)\frac{1}{2}\Delta h_{il} + (-2)\frac{1}{2}g^{jk_1}[\nabla_l\nabla_j h_{k_1i} + \nabla_i\nabla_j h_{k_1l} - \nabla_i\nabla_l h_{k_1j}] \\ &+ (-2)\frac{1}{2}g^{jk_1}[R^{k_2}_{jlk_1}h_{k_2i} + R^{i_1}_{jli}h_{k_1i_1} + R^{k_2}_{jik_1}h_{k_2l} + R^{l_1}_{jil}h_{k_1l_1}] \end{aligned}$$

$$\begin{aligned} &\frac{1}{2}g^{jk_1}[\nabla_l\nabla_j h_{k_1i} + \nabla_i\nabla_j h_{k_1l} - \nabla_i\nabla_l h_{k_1j}] \\ =&\frac{1}{2}g^{jk_1}[\frac{1}{2}\nabla_l(\nabla_j h_{k_1i} + \nabla_i h_{k_1j} - \nabla_{k_1}h_{ji}) + \frac{1}{2}\nabla_i(-\nabla_j h_{k_1l} + \nabla_l h_{k_1j} - \nabla_{k_1}h_{jl})] \end{aligned}$$

By the contraction of second Bianchi identity,

$$g^{jk_1}(\nabla_j R_{k_1i} + \nabla_i R_{k_1j} - \nabla_{k_1}R_{ji}) = 0$$

and $h = Ric + (n-1)g$, we have

$$\begin{aligned} \frac{d}{dt}Ric_{il} =& -(-2)\frac{1}{2}\Delta h_{il} + (-2)\frac{1}{2}g^{jk_1}[R^{k_2}_{jlk_1}h_{k_2i} + R^{i_1}_{jli}h_{k_1i_1} + R^{k_2}_{jik_1}h_{k_2l} + R^{l_1}_{jil}h_{k_1l_1}] \\ =& -(-2)\frac{1}{2}[\Delta h_{il} - g^{jk_1}R_{lj}h_{k_1i} - g^{jk_1}R_{ij}h_{k_1l} + 2g^{jk_1}g^{i_1i_2}R_{ii_2lj}h_{k_1i_1}] \\ =& -(-2)\frac{1}{2}\Delta_L h_{il} \end{aligned}$$

where Δ_L is the **Lichnerowicz Laplacian** operator defining as following

$$\Delta_L h_{il} = \Delta h_{il} - g^{jk_1}R_{lj}h_{k_1i} - g^{jk_1}R_{ij}h_{k_1l} + 2g^{jk_1}g^{i_1i_2}R_{ii_2lj}h_{k_1i_1}$$

By the fact that $R_{lj} = h_{lj} - (n-1)g_{lj}$, we have

$$\frac{d}{dt}R_{il} = -(-2)\frac{1}{2}[\Delta h_{il} - 2g^{jk_1}h_{lj}h_{k_1 i} + 2(n-1)h_{li} + 2g^{jk_1}g^{i_1 i_2}R_{ii_2 lj}h_{k_1 i_1}]$$

Therefore, we can get the curvature evolution equation, which is

$$\frac{d}{dt}h_{il} = \Delta h_{il} - 2g^{jk_1}h_{lj}h_{k_1 i} + 2g^{jk_1}g^{i_1 i_2}R_{ii_2 lj}h_{k_1 i_1}$$

or

$$\frac{d}{dt}h_{il} = \Delta_L h_{il} - 2(n-1)h_{il}$$

In particularly, for normalized Einstein manifold with negative Ricci curvature, we have

$$R_{ij} = -(n-1)g_{ij}$$

Therefore,

$$\Delta_L h_{il} = \Delta h_{il} + 2(n-1)h_{il} + 2g^{jk_1}g^{i_1 i_2}R_{ii_2 lj}h_{k_1 i_1}$$

Moreover, for the hyperbolic space with the sectional curvature equal to -1, we have

$$R_{ii_2 lj} = -(g_{il}g_{i_2 j} - g_{ij}g_{i_2 l})$$

Then, we have

$$\begin{aligned}\Delta_L h_{il} =& \Delta h_{il} + 2(n-1)h_{il} + 2g^{jk_1}g^{i_1 i_2}R_{ii_2 lj}h_{k_1 i_1} = \Delta h_{il} + 2(n-1)h_{il} + 2h_{il} \\ =& \Delta h_{il} + 2nh_{il}\end{aligned}$$

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